Preface

Welcome to *Intermediate Algebra*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax Resources

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Errata

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Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Intermediate Algebra*

Intermediate Algebra is designed to meet the scope and sequence requirements of a one-semester Intermediate algebra course. The book’s organization makes it easy to adapt to a variety of course syllabi. The text expands on the fundamental concepts of algebra while addressing the needs of students with diverse backgrounds and learning styles. Each topic builds upon previously developed material to demonstrate the cohesiveness and structure of mathematics.

Coverage and Scope

Intermediate Algebra continues the philosophies and pedagogical features of *Prealgebra* and *Elementary Algebra*, by Lynn Marecek and MaryAnne Anthony-Smith. By introducing the concepts and vocabulary of algebra in a nurturing, non-threatening environment while also addressing the needs of students with diverse backgrounds and learning styles, the book helps students gain confidence in their ability to succeed in the course and become successful college students.

The material is presented as a sequence of small, and clear steps to conceptual understanding. The order of topics was carefully planned to emphasize the logical progression throughout the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

Chapter 1: Foundations
Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, decimals and real numbers, to give the student a solid base that will support their study of algebra.

Chapter 2: Solving Linear Equations and Inequalities
In Chapter 2, students learn to solve linear equations using the Properties of Equality and a general strategy. They use a problem-solving strategy to solve number, percent, mixture and uniform motion applications. Solving a formula for a specific variable, and also solving both linear and compound inequalities is presented.

Chapter 3: Graphs and Functions
Chapter 3 covers the rectangular coordinate system where students learn to plot graph linear equations in two variables, graph with intercepts, understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities. The chapter also introduces relations and functions as well as graphing of functions.

Chapter 4: Systems of Linear Equations
Chapter 4 covers solving systems of equations by graphing, substitution, and elimination; solving applications with systems of equations, solving mixture applications with systems of equations, and graphing systems of linear inequalities. Systems of equations are also solved using matrices and determinants.

Chapter 5: Polynomials and Polynomial Functions
In Chapter 5, students learn how to add and subtract polynomials, use multiplication properties of exponents, multiply polynomials, use special products, divide monomials and polynomials, and understand integer exponents and scientific notation.
Chapter 6: Factoring
In Chapter 6, students learn the process of factoring expressions and see how factoring is used to solve quadratic equations.
Chapter 7: Rational Expressions and Functions
In Chapter 7, students work with rational expressions, solve rational equations and use them to solve problems in a variety of applications, and solve rational inequalities.
Chapter 8: Roots and Radical
In Chapter 8, students simplify radical expressions, rational exponents, perform operations on radical expressions, and solve radical equations. Radical functions and the complex number system are introduced
Chapter 9: Quadratic Equations
In Chapter 9, students use various methods to solve quadratic equations and equations in quadratic form and learn how to use them in applications. Students will graph quadratic functions using their properties and by transformations.
Chapter 10: Exponential and Logarithmic Functions
In Chapter 10, students find composite and inverse functions, evaluate, graph, and solve both exponential and logarithmic functions.
Chapter 11: Conics
In Chapter 11, the properties and graphs of circles, parabolas, ellipses and hyperbolas are presented. Students also solve applications using the conics and solve systems of nonlinear equations.
Chapter 12: Sequences, Series and the Binomial Theorem
In Chapter 12, students are introduced to sequences, arithmetic sequences, geometric sequences and series and the binomial theorem.

All chapters are broken down into multiple sections, the titles of which can be viewed in the **Table of Contents**.

Key Features and Boxes

Examples Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. Typically, we include multiple examples for each learning objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

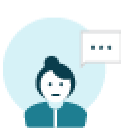
All examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select examples), we show students how to check the solution. Most examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors "talk through" examples as they write on the board in class.

Be Prepared! Each section, beginning with Section 2.1, starts with a few "Be Prepared!" exercises so that students can determine if they have mastered the prerequisite skills for the section. Reference is made to specific Examples from previous sections so students who need further review can easily find explanations. Answers to these exercises can be found in the supplemental resources that accompany this title.

Try It

Try it The Try It feature includes a pair of exercises that immediately follow an Example, providing the student with an immediate opportunity to solve a similar problem with an easy reference to the example. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try It exercises are located in the Answer Key.

How To



How To Examples use a three column format to demonstrate how to solve an example with a certain procedure. The first column states the formal step, the second column is in words as the teacher would explain the process, and then the third column is the actual math. A How To procedure box follows each of these How To examples and summarizes the series of steps from the example. These procedure boxes provide an easy reference for students.

Media

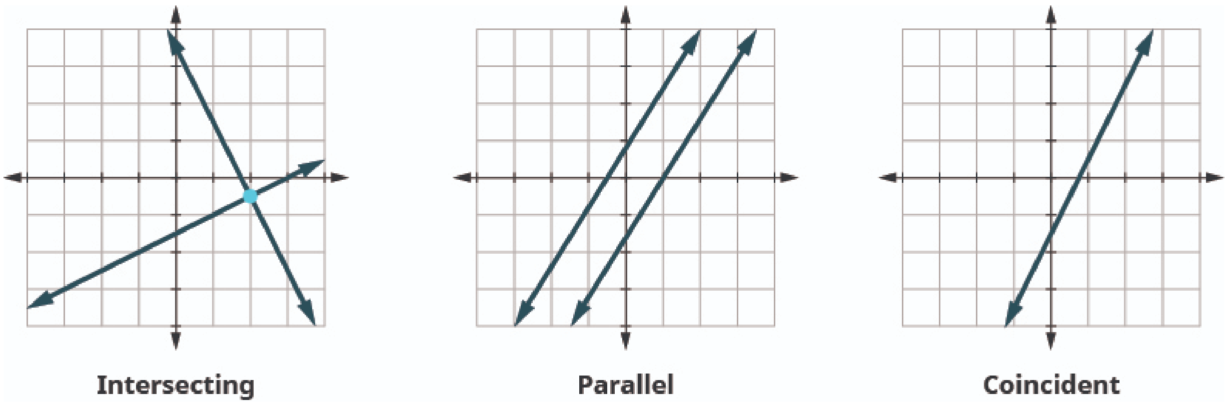
Media The “Media” icon appears at the conclusion of each section, just prior to the Self Check. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *Intermediate Algebra*.

Self Check The Self Check includes the learning objectives for the section so that students can self-assess their mastery and make concrete plans to improve.

Art Program

Intermediate Algebra contains many figures and illustrations. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions.



Section Exercises and Chapter Review

Section Exercises Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. Exercise sets are named *Practice Makes Perfect* to encourage completion of homework assignments.

- Exercises correlate to the learning objectives. This facilitates assignment of personalized study plans based on individual student needs.
- Exercises are carefully sequenced to promote building of skills.
- Values for constants and coefficients were chosen to practice and reinforce arithmetic facts.
- Even and odd-numbered exercises are paired.
- Exercises parallel and extend the text examples and use the same instructions as the examples to help students easily recognize the connection.
- Applications are drawn from many everyday experiences, as well as those traditionally found in college math texts.
- **Everyday Math** highlights practical situations using the concepts from that particular section
- **Writing Exercises** are included in every exercise set to encourage conceptual understanding, critical thinking, and literacy.

Chapter review Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students

can use to prepare for exams.

- **Key Terms** provide a formal definition for each bold-faced term in the chapter.
- **Key Concepts** summarize the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include practice problems that recall the most important concepts from each section.
- **Practice Test** includes additional problems assessing the most important learning objectives from the chapter.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, manipulative mathematics worksheets, an answer key to the Be Prepared Exercises, and an answer guide to the section review exercises. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

OpenStax partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on openstax.org.

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Lynn Marecek has been teaching mathematics at Santa Ana College for many years has focused her career on meeting the needs of developmental math students. At Santa Ana College, she has been awarded the Distinguished Faculty Award, Innovation Award, and the Curriculum Development Award four times. She is a Coordinator of the Freshman Experience Program, the Department Facilitator for Redesign, and a member of the Student Success and Equity Committee, and the Basic Skills Initiative Task Force.

She is the coauthor with MaryAnne Anthony-Smith of *Strategies for Success: Study Skills for the College Math Student*, *Prealgebra* published by OpenStax and *Elementary Algebra* published by OpenStax.

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Relations and Functions
By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Note:
Before you get started, take this readiness quiz.

1. Evaluate $3x-5$ when $x=-2$.
   If you missed this problem, review [link].
2. Evaluate $2x^2-x-3$ when $x=a$.
   If you missed this problem, review [link].
3. Simplify: $7x-1-4x+5$.
   If you missed this problem, review [link].

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.
Equation:

$$(\text{Student name, Student ID \#})$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Note:
Relation

A **relation** is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the **domain**. All the *y*-values in the ordered pairs together make up the **range**.

Example:
Exercise:

Problem: For the relation $\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution:

$$\{(1,1),(2,4),(3,9),(4,16),(5,25)\}$$

ⓐ The domain is the set of all *x*-values of the relation. $\{1,2,3,4,5\}$

ⓑ The range is the set of all *y*-values of the relation. $\{1,4,9,16,25\}$

Note:
Exercise:

Problem: For the relation $\{(1,1),(2,8),(3,27),(4,64),(5,125)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution:

ⓐ $\{1,2,3,4,5\}$
ⓑ $\{1,8,27,64,125\}$

Note:
Exercise:

Problem: For the relation $\{(1,3),(2,6),(3,9),(4,12),(5,15)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution:

ⓐ $\{1, 2, 3, 4, 5\}$
ⓑ $\{3, 6, 9, 12, 15\}$

Note:
Mapping
A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

Example:
Exercise:

Problem:

Use the **mapping** of the relation shown to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

Solution:

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person’s name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all *x*-values of the relation.

{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

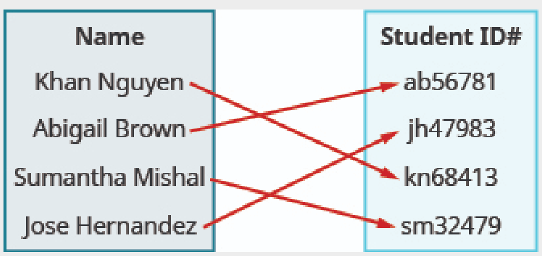
ⓒ The range is the set of all *y*-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

Note:
Exercise:

Problem:

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



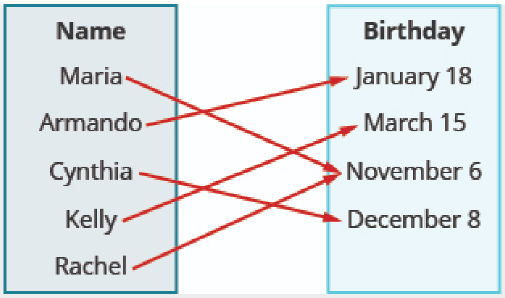
Solution:

ⓐ (Khanh Nguyen, kn68413), (Abigail Brown, ab56781), (Sumantha Mishal, sm32479), (Jose Hern and ez, jh47983) ⓑ {Khanh Nguyen, Abigail Brown, Sumantha Mishal, Jose Hern and ez} ⓒ {kn68413, ab56781, sm32479, jh47983}

Note:
Exercise:

Problem:

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

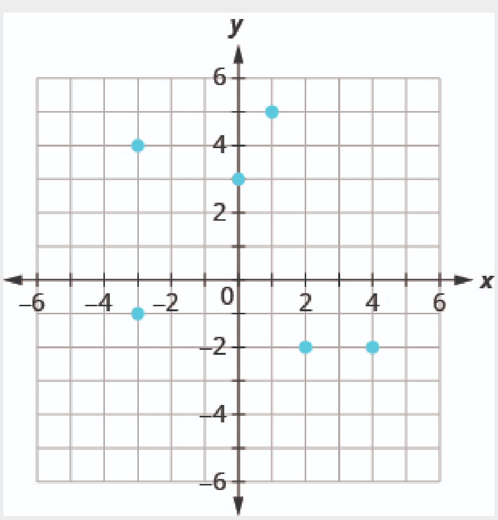
ⓐ (Maria, November 6), (Arm and o, January 18), (Cynthia, December 8), (Kelly, March 15), (Rachel, November 6) ⓑ {Maria, Arm and o, Cynthia, Kelly, Rachel} ⓒ {November 6, January 18, December 8, March 15}

A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all *x*-coordinates is the domain of the relation and the set of all *y*-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

Example:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ The ordered pairs of the relation are:
$$\{(1,5),(-3,-1),(4,-2),(0,3),(2,-2),(-3,4)\}.$$

ⓑ The domain is the set of all *x*-values of the relation: $\{-3,0,1,2,4\}.$

Notice that while -3 repeats, it is only listed once.

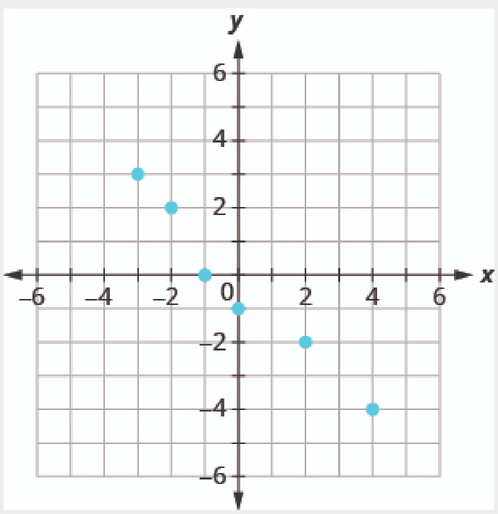
ⓒ The range is the set of all *y*-values of the relation: $\{-2,-1,3,4,5\}.$

Notice that while -2 repeats, it is only listed once.

Note:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



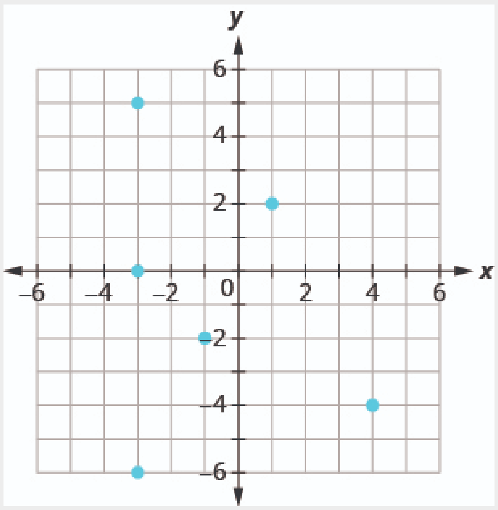
Solution:

ⓐ $(-3,3),(-2,2),(-1,0),$
$(0,-1),(2,-2),(4,-4)$
ⓑ $\{-3,-2,-1,0,2,4\}$
ⓒ$\{3,2,0,-1,-2,-4\}$

Note:
Exercise:

Problem:

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ $(-3, 0), (-3, 5), (-3, -6),$
$(-1, -2), (1, 2), (4, -4)$
ⓑ $\{-3, -1, 1, 4\}$
ⓒ $\{-6, 0, 5, -2, 2, -4\}$

Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Note:
Function
A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from [link] helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in [link] is a function.

The relation shown by the graph in [link] includes the ordered pairs $(-3,-1)$ and $(-3,4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

Example:
Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Solution:

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

(i) Each *x*-value is matched with only one *y*-value. So this relation is a function.

(ii) The domain is the set of all *x*-values in the relation.
The domain is: $\{-3,-2,-1,0,1,2,3\}$.

(iii) The range is the set of all *y*-values in the relation. Notice we do not list range values twice.
The range is: $\{27,8,1,0\}$.

ⓑ $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

(i) The *x*-value 9 is matched with two *y*-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all *x*-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0,1,2,4,9\}$.

(iii) The range is the set of all *y*-values in the relation.
The range is: $\{-3,-2,-1,0,1,2,3\}$.

Note:
Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

ⓐ $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$

ⓑ $\{(8,-4),(4,-2),(2,-1),(0,0),(2,1),(4,2),(8,4)\}$

Solution:

ⓐ Yes; $\{-3,-2,-1,0,1,2,3\}$;
$\{-6,-4,-2,0,2,4,6\}$
ⓑ No; $\{0,2,4,8\}$;
$\{-4,-2,-1,0,1,2,4\}$

Note:
Exercise:

Problem:

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(27,-3),(8,-2),(1,-1),(0,0),(1,1),(8,2),(27,3)\}$

ⓑ $\{(7,-3),(-5,-4),(8,-0),(0,0),(-6,4),(-2,2),(-1,3)\}$

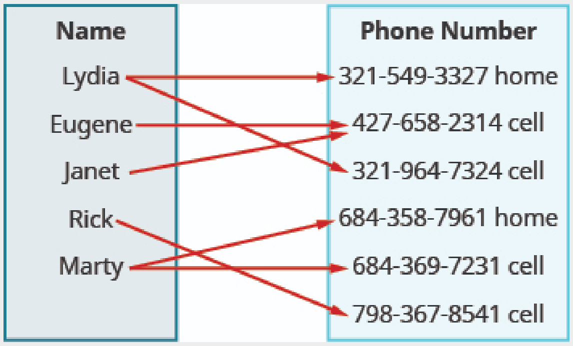
Solution:

ⓐ No; $\{0,1,8,27\}$;
$\{-3,-2,-1,0,2,2,3\}$
ⓑ Yes; $\{7,-5,8,0,-6,-2,-1\}$;
$\{-3,-4,0,4,2,3\}$

Example:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ Both Lydia and Marty have two phone numbers. So each *x*-value is not matched with only one *y*-value. So this relation is not a function.

ⓑ The domain is the set of all *x*-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

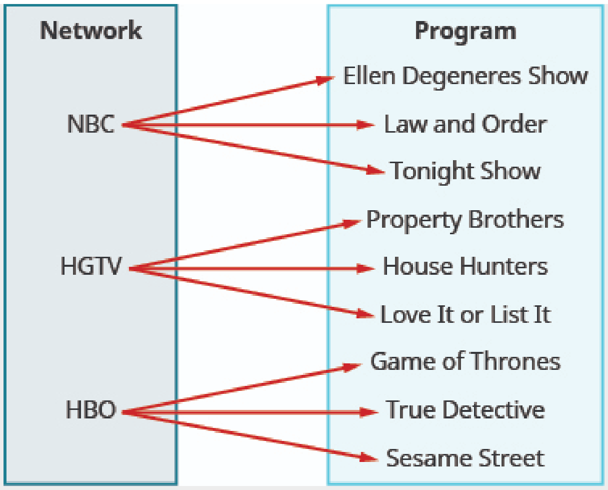
ⓒ The range is the set of all *y*-values in the relation. The range is:

$\{321\text{-}549\text{-}3327, 427\text{-}658\text{-}2314, 321\text{-}964\text{-}7324, 684\text{-}358\text{-}7961, 684\text{-}369\text{-}7231, 798\text{-}367\text{-}8541\}$

Note:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



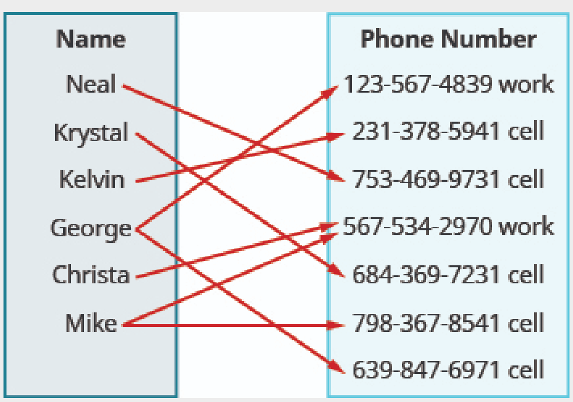
Solution:

ⓐ no ⓑ {NBC, HGTV, HBO} ⓒ {Ellen Degeneres Show, Law and Order, Tonight Show, Property Brothers, House Hunters, Love it or List it, Game of Thrones, True Detective, Sesame Street}

Note:
Exercise:

Problem:

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution:

ⓐ No ⓑ {Neal, Krystal, Kelvin, George, Christa, Mike} ⓒ {123-567-4839 work, 231-378-5941 cell, 743-469-9731 cell, 567-534-2970 work, 684-369-7231 cell, 798-367-

8541 cell, 639-847-6971 cell}

In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

Example:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $2x+y=7$ ⓑ $y=x^2+1$ ⓒ$x+y^2=3$

Solution:

ⓐ $2x+y=7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y = -2x + 7$
For example, if $x=3$:	$y = -2 \cdot 3 + 7$
	$y = 1$

We have that when $x=3$, then $y=1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y=x^2+1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y = x^2 + 1$
For example, if $x = 2$:	$y = 2^2 + 1$
	$y = 5$

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

ⓒ

	$x + y^2 = 3$
Isolate the y term.	$y^2 = -x + 3$
Let's substitute $x = 2$.	$y^2 = -2 + 3$
	$y^2 = 1$
This give us two values for y.	$y = 1$ $y = -1$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

Note:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $4x+y=-3$ ⓑ $x+y^2=1$ ⓒ $y-x^2=2$

Solution:

ⓐ yes ⓑ no ⓒ yes

Note:
Exercise:

Problem: Determine whether each equation is a function.

ⓐ $x+y^2=4$ ⓑ $y=x^2-7$ ⓒ $y=5x-4$

Solution:

ⓐ no ⓑ yes ⓒ yes

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function *f*, we write this range value *y* as $f\left(x\right)$. This is called function notation and is read *f* of *x* or the value of *f* at *x*. In this case the parentheses does not indicate multiplication.

Note:
Function Notation
For the function $y=f(x)$
Equation:

$$\begin{array}{l} f \text{ is the name of the function} \\ x \text{ is the domain value} \\ f(x) \text{ is the range value } y \text{ corresponding to the value } x \end{array}$$

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

Note:
Independent and Dependent Variables
For the function $y = f(x)$,
Equation:

$$\begin{array}{l}x\text{ is the independent variable as it can be any value in the domain}\\ y\text{ the dependent variable as its value depends on }x\end{array}$$

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5$. To find the value of *y* when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2$.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f\left(x\right) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let $x = 2$.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

Example:
Exercise:

Problem: For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(a)$

Solution:

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3)$, substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$

	$f(3) = 26$

ⓑ

		$f(x) = 2x^2 + 3x - 1$
To evaluate $f(-2)$, substitute -2 for x.		$f(-2) = 2(-2)^2 + 3(-2) - 1$
Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

ⓒ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(a)$, substitute a for x.	$f(a) = 2(a)^2 + 3 \cdot a - 1$
Simplify.	$f(a) = 2a^2 + 3a - 1$

Note:
Exercise:

Problem: For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-1)$ ⓒ $f(t)$

Solution:

ⓐ $f(3) = 22$ ⓑ $f(-1) = 6$ ⓒ $f(t) = 3t^2 - 2t - 1$

Note:
Exercise:

Problem: For the function $f(x) = 2x^2 + 4x - 3$, evaluate the function.

ⓐ $f(2)$ ⓑ $f(-3)$ ⓒ $f(h)$

Solution:

ⓐ $(2) = 13$ ⓑ $f(-3) = 3$
ⓒ $f(h) = 2h^2 + 4h - 3$

In the last example, we found $f(x)$ for a constant value of *x*. In the next example, we are asked to find $g(x)$ with values of *x* that are variables. We still follow the same procedure and substitute the variables in for the *x*.

Example:
Exercise:

Problem: For the function $g(x) = 3x - 5$, evaluate the function.

ⓐ $g(h^2)$ ⓑ $g(x + 2)$ ⓒ $g(x) + g(2)$

Solution:

ⓐ

			$g(x)=3x-5$
To evaluate $g\left(h^2\right)$, substitute h^2 for x.			$g(h^2)=3h^2-5$
			$g(h^2)=3h^2-5$

ⓑ

	$g(x)=3x-5$
To evaluate $g\left(x+2\right)$, substitute $x+2$ for x.	$g(x+2)=3(x+2)-5$
Simplify.	$g(x+2)=3x+6-5$
	$g(x+2)=3x+1$

ⓒ

		$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.		$g(2) = 3 \cdot 2 - 5$
		$g(2) = 1$
Now find $g(x) + g(2)$		$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$
Simplify.		$g(x) + g(2) = 3x - 5 + 1$
		$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x + 1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

Note:
Exercise:

Problem: For the function $g(x) = 4x - 7$, evaluate the function.

ⓐ $g(m^2)$ ⓑ $g(x-3)$ ⓒ $g(x) - g(3)$

Solution:

ⓐ $4m^2 - 7$ ⓑ $4x - 19$
ⓒ $x - 12$

Note:
Exercise:

Problem: For the function $h(x) = 2x + 1$, evaluate the function.

ⓐ $h(k^2)$ ⓑ $h(x+1)$ ⓒ $h(x)+h(1)$

Solution:

ⓐ $2k^2+1$ ⓑ $2x+3$
ⓒ $2x+4$

Many everyday situations can be modeled using functions.

Example:
Exercise:

Problem:

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t)=75+10t$ represents the relation between the number of emails, N, and the time, t, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(5)$. Explain what this result means.

Solution:

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, N, depends on the number of days, t. Therefore, the variable N, is the dependent variable and the variable t is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t)=75+10t$
Substitute in $t=5$.	$N(5)=75+10\cdot5$
Simplify.	$N(5)=75+50$

	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Note:
Exercise:

Problem:

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, N, and the time, t, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(7)$. Explain what this result means.

Solution:

ⓐ t IND; N DEP ⓑ 205; the number of unread emails in Bryan's account on the seventh day.

Note:
Exercise:

Problem:

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, N, and the time, t, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(14)$. Explain what this result means.

Solution:

ⓐ t IND; N DEP ⓑ 460; the number of unread emails in Anthony's account on the fourteenth day

Note:
Access this online resource for additional instruction and practice with relations and functions.

- Introduction to Functions

Key Concepts

- **Function Notation:** For the function $y = f(x)$
  - *f* is the name of the function
  - *x* is the domain value
  - $f(x)$ is the range value *y* corresponding to the value *x*
    We read $f(x)$ as *f* of *x* or the value of *f* at *x*.
- **Independent and Dependent Variables:** For the function $y = f(x)$,
  - *x* is the independent variable as it can be any value in the domain
  - *y* is the dependent variable as its value depends on *x*

Practice Makes Perfect

Find the Domain and Range of a Relation

In the following exercises, for each relation ⓐ find the domain of the relation ⓑ find the range of the relation.

Exercise:

Problem: $\{(1,4),(2,8),(3,12),(4,16),(5,20)\}$

Solution:

ⓐ {1, 2, 3, 4, 5} ⓑ {4, 8, 12, 16, 20}

Exercise:

Problem: $\{(1,-2),(2,-4),(3,-6),(4,-8),(5,-10)\}$

Exercise:

Problem: $\{(1,7),(5,3),(7,9),(-2,-3),(-2,8)\}$

Solution:

ⓐ {1, 5, 7, −2} ⓑ {7, 3, 9, −3, 8}

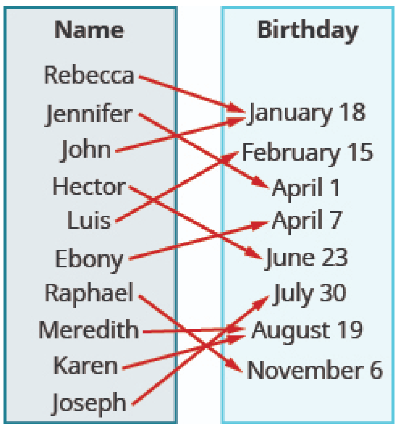
Exercise:

Problem: $\{(11, 3), (-2, -7), (4, -8), (4, 17), (-6, 9)\}$

In the following exercises, use the mapping of the relation to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

Exercise:

Problem:

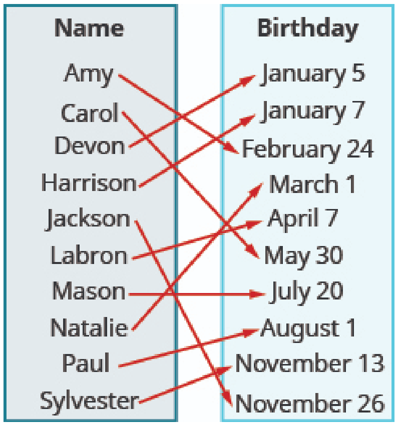


Solution:

ⓐ (Rebecca, January 18), (Jennifer, April 1), (John, January 18), (Hector, June 23), (Luis, February 15), (Ebony, April 7), (Raphael, November 6), (Meredith, August 19), (Karen, August 19), (Joseph, July 30)
ⓑ {Rebecca, Jennifer, John, Hector, Luis, Ebony, Raphael, Meredith, Karen, Joseph}
ⓒ {January 18, April 1, June 23, February 15, April 7, November 6, August 19, July 30}

Exercise:

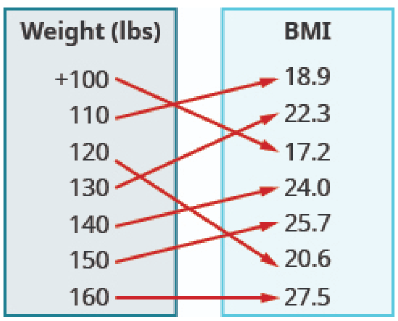
Problem:



Exercise:

Problem:

For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$ is considered healthy.



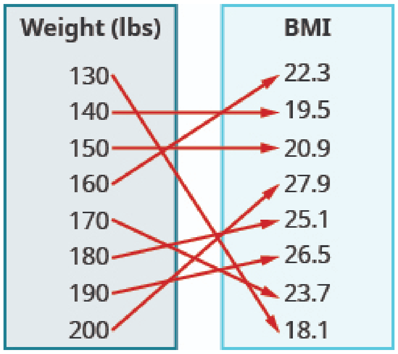
Solution:

ⓐ (+100, 17. 2), (110, 18.9), (120, 20.6), (130, 22.3), (140, 24.0), (150, 25.7), (160, 27.5)
ⓑ {+100, 110, 120, 130, 140, 150, 160,} ⓒ {17.2, 18.9, 20.6, 22.3, 24.0, 25.7, 27.5}

Exercise:

Problem:

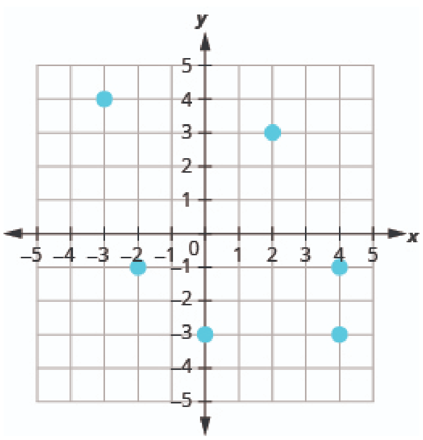
For a man of height $5'11''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$ is considered healthy.



In the following exercises, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:

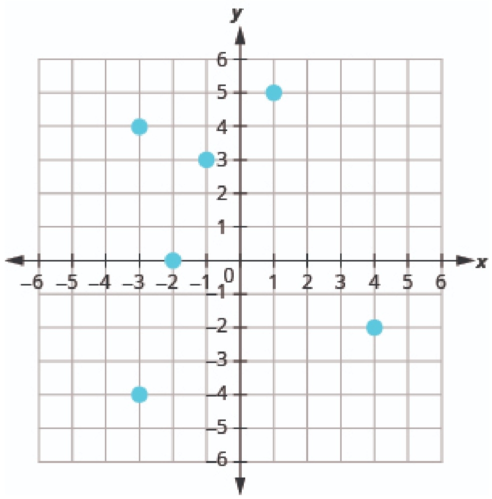


Solution:

ⓐ (2, 3), (4, −3), (−2, −1), (−3, 4), (4, −1), (0, −3) ⓑ {−3, −2, 0, 2, 4}
ⓒ {−3, −1, 3, 4}

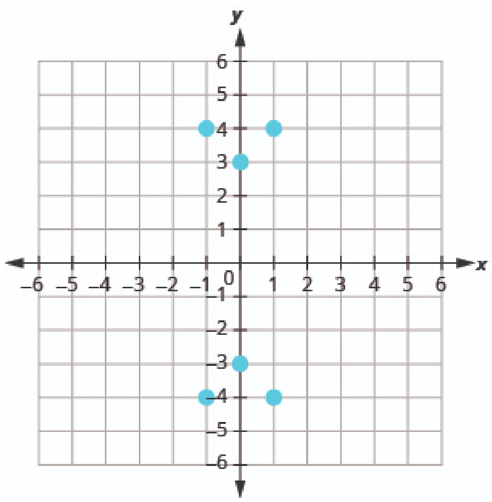
Exercise:

Problem:



Exercise:

Problem:

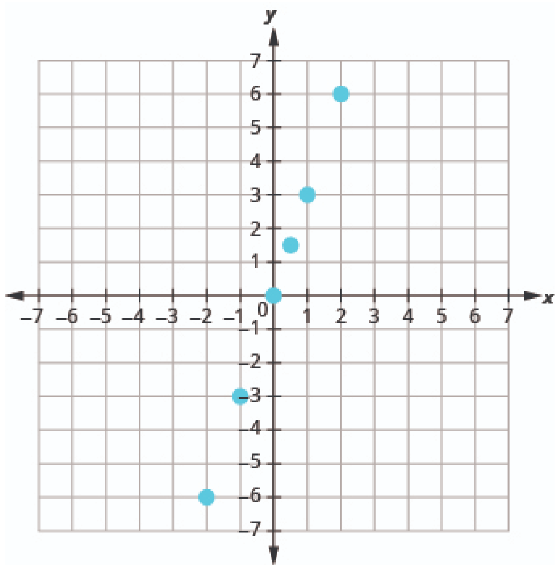


Solution:

ⓐ (1, 4), (1, −4), (−1, 4), (−1, −4), (0, 3), (0, −3) ⓑ {−1, 0, 1} ⓒ {−4, −3, 3,4}

Exercise:

Problem:



Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

Exercise:

Problem: $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {9, 4, 1, 0}

Exercise:

Problem: $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Exercise:

Problem: $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ 0, 1, 8, 27}

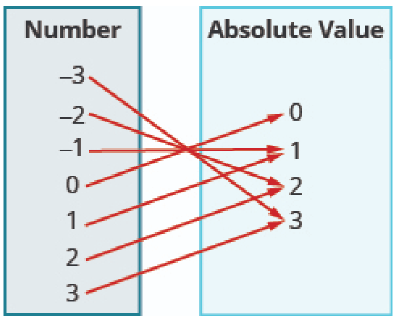
Exercise:

Problem: $\{(-3,-27),(-2,-8),(-1,-1),$
$(0,0),(1,1),(2,8),(3,27)\}$

In the following exercises, use the mapping to ⓐ determine whether the relation is a function, ⓑ find the domain of the function, and ⓒ find the range of the function.

Exercise:

Problem:

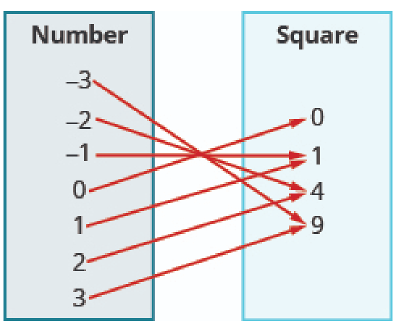


Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3} ⓒ {0, 1, 2, 3}

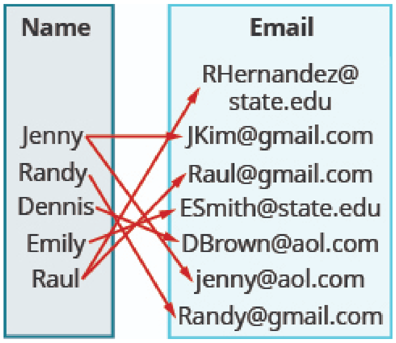
Exercise:

Problem:



Exercise:

Problem:

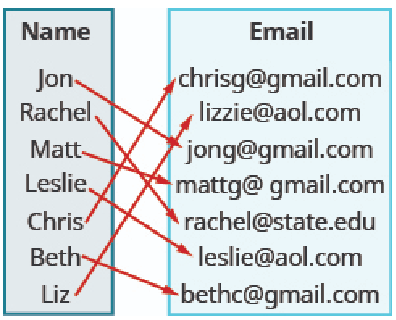


Solution:

ⓐ no ⓑ {Jenny, R and y, Dennis, Emily, Raul} ⓒ {RHern and ez@state.edu, JKim@gmail.com, Raul@gmail.com, ESmith@state.edu, DBroen@aol.com, jenny@aol.cvom, R and y@gmail.com}

Exercise:

Problem:



In the following exercises, determine whether each equation is a function.

Exercise:

Problem: ⓐ $2x + y = -3$
ⓑ $y = x^2$
ⓒ$x + y^2 = -5$

Solution:

ⓐ yes ⓑ yes ⓒ no

Exercise:

ⓐ $y = 3x - 5$
ⓑ $y = x^3$
Problem: ⓒ$2x + y^2 = 4$

Exercise:

ⓐ $y - 3x^3 = 2$
ⓑ $x + y^2 = 3$
Problem: ⓒ$3x - 2y = 6$

Solution:

ⓐ yes ⓑ no ⓒ yes

Exercise:

ⓐ $2x - 4y = 8$
ⓑ $-4 = x^2 - y$
Problem: ⓒ$y^2 = -x + 5$

Find the Value of a Function

In the following exercises, evaluate the function: ⓐ $f(2)$ ⓑ $f(-1)$ ⓒ $f(a)$.

Exercise:

Problem: $f(x) = 5x - 3$

Solution:

ⓐ $f(2) = 7$ ⓑ $f(-1) = -8$ ⓒ $f(a) = 5a - 3$

Exercise:

Problem: $f(x) = 3x + 4$

Exercise:

Problem: $f(x) = -4x + 2$

Solution:

ⓐ $f(2) = -6$ ⓑ $f(-1) = 6$ ⓒ $f(a) = -4a + 2$

Exercise:

Problem: $f(x) = -6x - 3$

Exercise:

Problem: $f(x) = x^2 - x + 3$

Solution:

ⓐ $f(2) = 5$ ⓑ $f(-1) = 5$
ⓒ $f(a) = a^2 - a + 3$

Exercise:

Problem: $f(x) = x^2 + x - 2$

Exercise:

Problem: $f(x) = 2x^2 - x + 3$

Solution:

ⓐ $f(2) = 9$ ⓑ $f(-1) = 6$
ⓒ $f(a) = 2a^2 - a + 3$

Exercise:

Problem: $f(x) = 3x^2 + x - 2$

In the following exercises, evaluate the function: ⓐ $g(h^2)$ ⓑ $g(x+2)$ ⓒ $g(x) + g(2)$.

Exercise:

Problem: $g(x) = 2x + 1$

Solution:

ⓐ $g(h^2) = 2h^2 + 1$
ⓑ $g(x+2) = 4x + 5$
ⓒ $g(x) + g(2) = 2x + 6$

Exercise:

Problem: $g(x) = 5x - 8$

Exercise:

Problem: $g(x) = -3x - 2$

Solution:

ⓐ $g(h^2) = -3h^2 - 2$
ⓑ $g\,(x+2) = -3x - 8$
ⓒ $g\,(x) + g\,(2) = -3x - 10$

Exercise:

Problem: $g(x) = -8x + 2$

Exercise:

Problem: $g(x) = 3 - x$

Solution:

ⓐ $g(h^2) = 3 - h^2$
ⓑ $g\,(x+2) = 1 - x$
ⓒ $g\,(x) + g\,(2) = 4 - x$

Exercise:

Problem: $g(x) = 7 - 5x$

In the following exercises, evaluate the function.

Exercise:

Problem: $f(x) = 3x^2 - 5x;\ f(2)$

Solution:

2

Exercise:

Problem: $g(x) = 4x^2 - 3x;\ g(3)$

Exercise:

Problem: $F(x) = 2x^2 - 3x + 1;$ $F(-1)$

Solution:

6

Exercise:

Problem: $G(x) = 3x^2 - 5x + 2;$ $G(-2)$

Exercise:

Problem: $h(t) = 2\left|t - 5\right| + 4;$ $f(-4)$

Solution:

22

Exercise:

Problem: $h(y) = 3\left|y - 1\right| - 3;$ $h(-4)$

Exercise:

Problem: $f(x) = \frac{x+2}{x-1};$ $f(2)$

Solution:

4

Exercise:

Problem: $g(x) = \frac{x-2}{x+2};$ $g(4)$

In the following exercises, solve.

Exercise:

Problem:

The number of unwatched shows in Sylvia's DVR is 85. This number grows by 20 unwatched shows per week. The function $N\left(t\right) = 85 + 20t$ represents the relation between the number of unwatched shows, *N*, and the time, *t*, measured in weeks.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N\left(4\right)$. Explain what this result means

Solution:

ⓐ *t* IND; *N* DEP
ⓑ $N\left(4\right) = 165$ the number of unwatched shows in Sylvia's DVR at the fourth week.

Exercise:

Problem:

Every day a new puzzle is downloaded into Ken's account. Right now he has 43 puzzles in his account. The function $N(t) = 43 + t$ represents the relation between the number of puzzles, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(30)$. Explain what this result means.

Exercise:

Problem:

The daily cost to the printing company to print a book is modeled by the function $C(x) = 3.25x + 1500$ where *C* is the total daily cost and *x* is the number of books printed.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(0)$. Explain what this result means.

ⓒ Find $N(1000)$. Explain what this result means.

Solution:

ⓐ *x* IND; *C* DEP
ⓑ $N(0) = 1500$ the daily cost if no books are printed
ⓒ $N(1000) = 4750$ the daily cost of printing 1000 books

Exercise:

Problem:

The daily cost to the manufacturing company is modeled by the function $C(x) = 7.25x + 2500$ where $C(x)$ is the total daily cost and *x* is the number of items manufactured.

ⓐ Determine the independent and dependent variable.

ⓑ Find $C(0)$. Explain what this result means.

ⓒ Find $C(1000)$. Explain what this result means.

Writing Exercises

Exercise:

Problem: In your own words, explain the difference between a relation and a function.

Exercise:

Problem: In your own words, explain what is meant by domain and range.

Exercise:

Problem: Is every relation a function? Is every function a relation?

Exercise:

Problem: How do you find the value of a function?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the domain and range of a relation.			
determine if a relation is a function.			
find the value of a function.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

domain of a relation
: The domain of a relation is all the *x*-values in the ordered pairs of the relation.

function
: A function is a relation that assigns to each element in its domain exactly one element in the range.

mapping
: A mapping is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

range of a relation

The range of a relation is all the *y*-values in the ordered pairs of the relation.

relation

A relation is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the domain. All the *y*-values in the ordered pairs together make up the range.

Graphs of Functions
By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

Note:
Before you get started, take this readiness quiz.

1. Evaluate: ⓐ 2^3 ⓑ 3^2.
   If you missed this problem, review [link].
2. Evaluate: ⓐ $|7|$ ⓑ $|-3|$.
   If you missed this problem, review [link].
3. Evaluate: ⓐ $\sqrt{4}$ ⓑ $\sqrt{16}$.
   If you missed this problem, review [link].

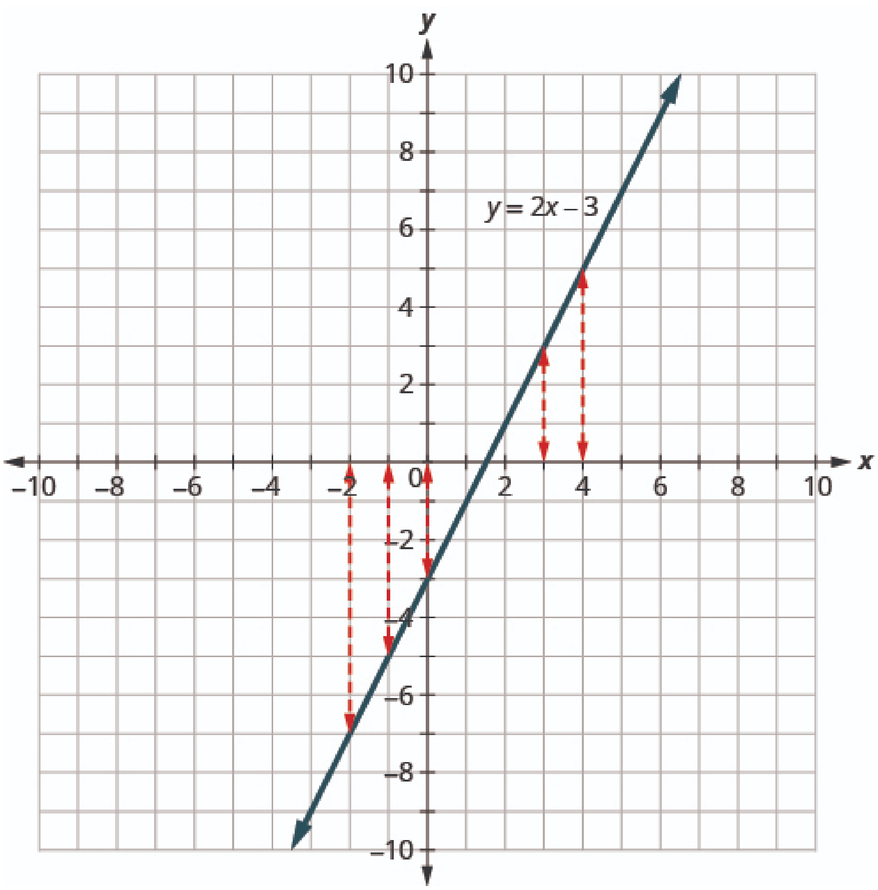
Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In [link], we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.



$y = 2x - 3$		
x	*y*	(*x*, *y*)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value.

If the vertical line hit the graph twice, the *x*-value would be mapped to two *y*-values, and so the graph would not represent a function.

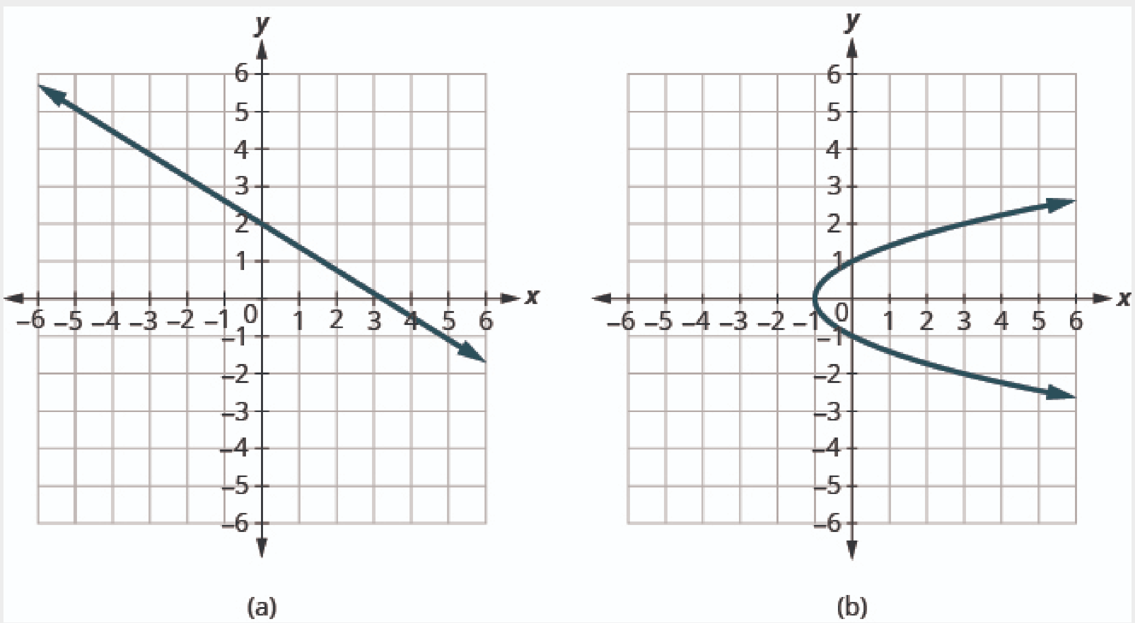
This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects

the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

Note:
Vertical Line Test
A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
If any vertical line intersects the graph in more than one point, the graph does not represent a function.

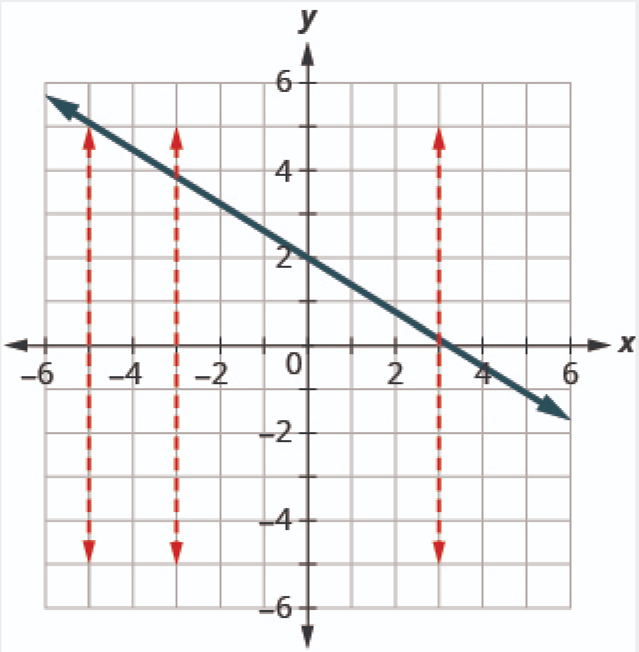
Example:
Exercise:

Problem: Determine whether each graph is the graph of a function.

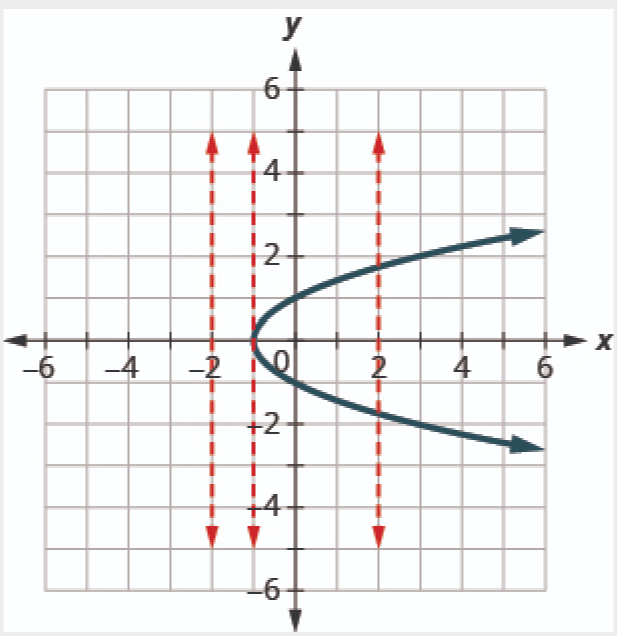


Solution:

ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.

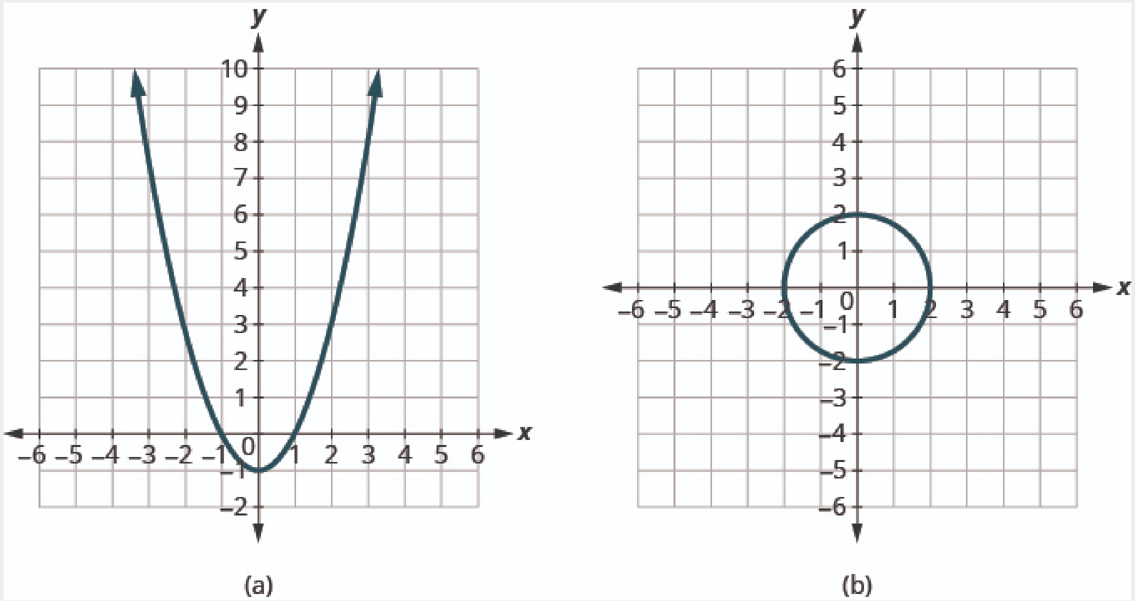


ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



Note:
Exercise:

Problem: Determine whether each graph is the graph of a function.

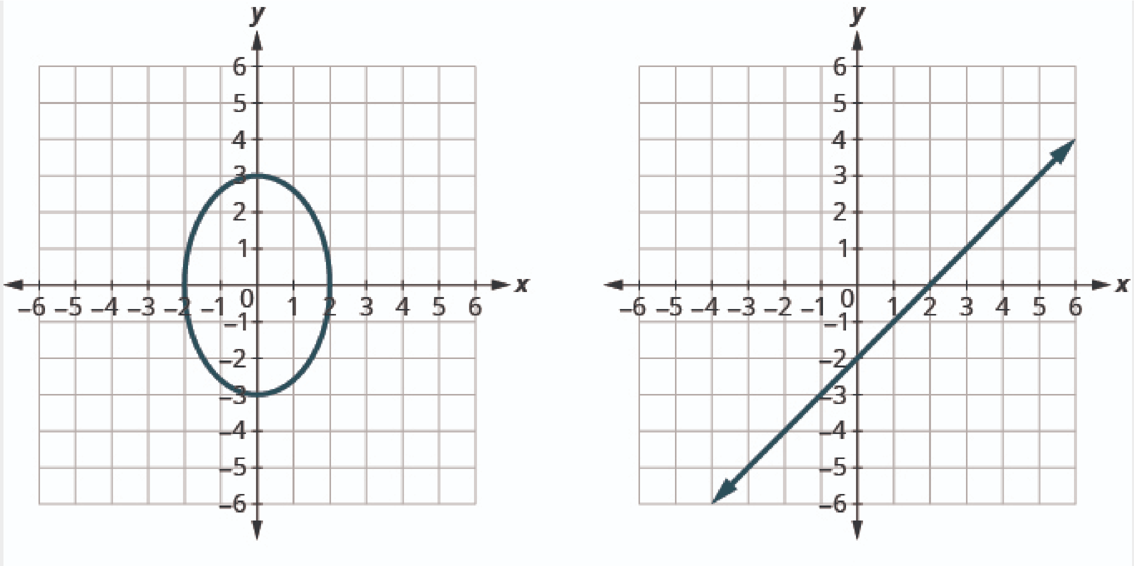


Solution:

ⓐ yes ⓑ no

Note:
Exercise:

Problem: Determine whether each graph is the graph of a function.



(a) (b)

Solution:

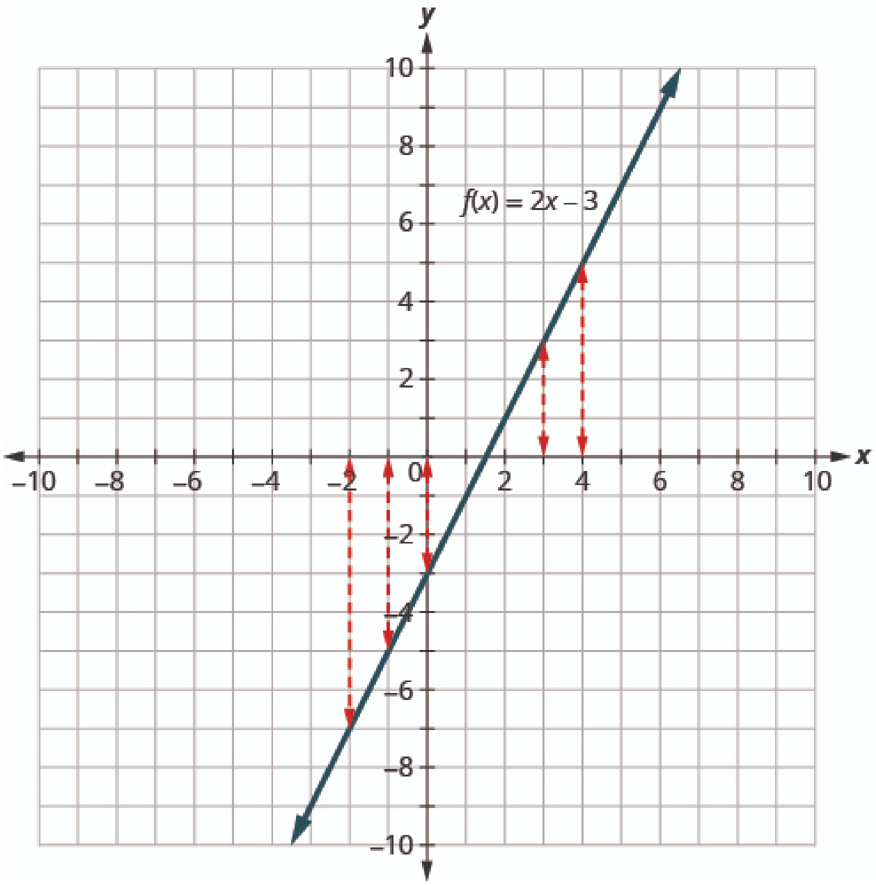
ⓐ no ⓑ yes

Identify Graphs of Basic Functions

We used the equation $y = 2x - 3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y = 2x - 3$ is a function.

We can write this as in function notation as $f\left(x\right) = 2x - 3$. It still means the same thing. The graph of the function is the graph of all ordered pairs $\left(x, y\right)$ where $y = f\left(x\right)$. So we can write the ordered pairs as $\left(x, f\left(x\right)\right)$. It looks different but the graph will be the same.

Compare the graph of $y = 2x - 3$ previously shown in [link] with the graph of $f\left(x\right) = 2x - 3$ shown in [link]. Nothing has changed but the notation.



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Note:

Graph of a Function

The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

Equation:

$$\begin{array}{rl} f & \text{name of function} \\ x & x\text{-coordinate of the ordered pair} \\ f(x) & y\text{-coordinate of the ordered pair} \end{array}$$

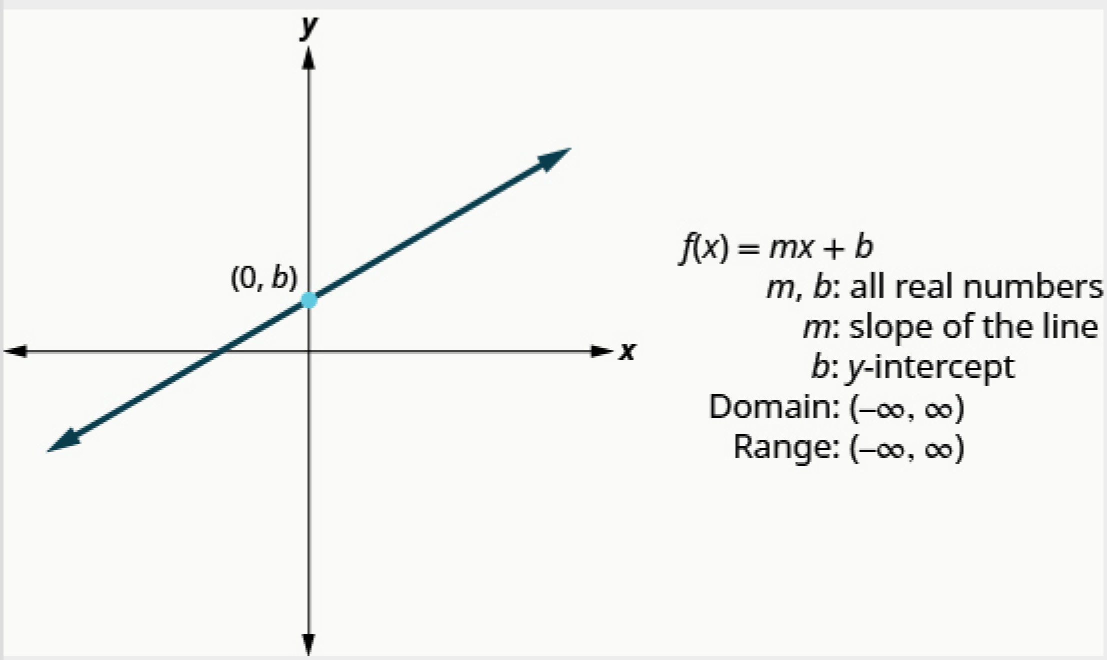
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f\left(x\right) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

Note:
Linear Function



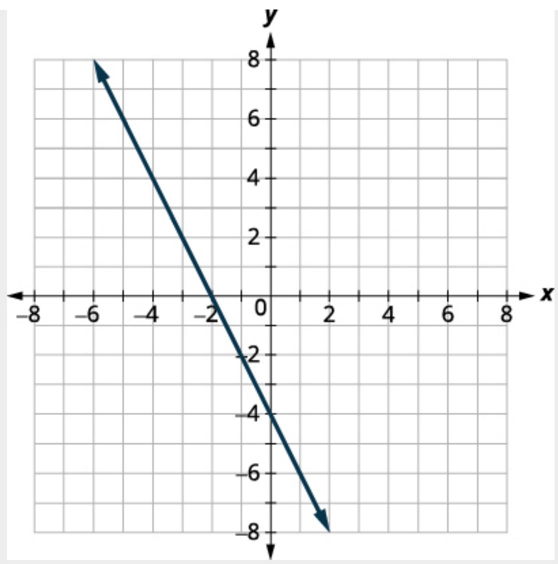
We will use the graphing techniques we used earlier, to graph the basic functions.

Example:
Exercise:

Problem: Graph: $f(x) = -2x - 4$.

Solution:

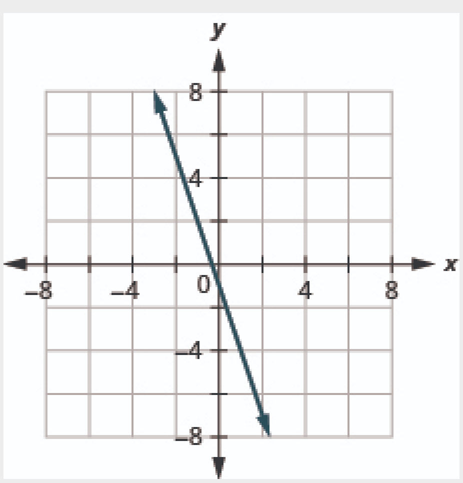
	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and *y*-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	



Note:
Exercise:

Problem: Graph: $f\left(x\right)=-3x-1$

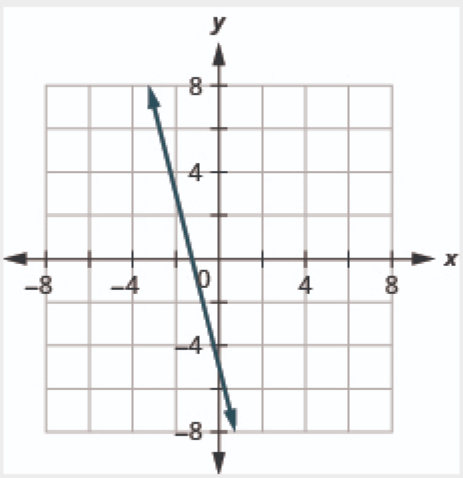
Solution:



Note:
Exercise:

Problem: Graph: $f\left(x\right)=-4x-5$

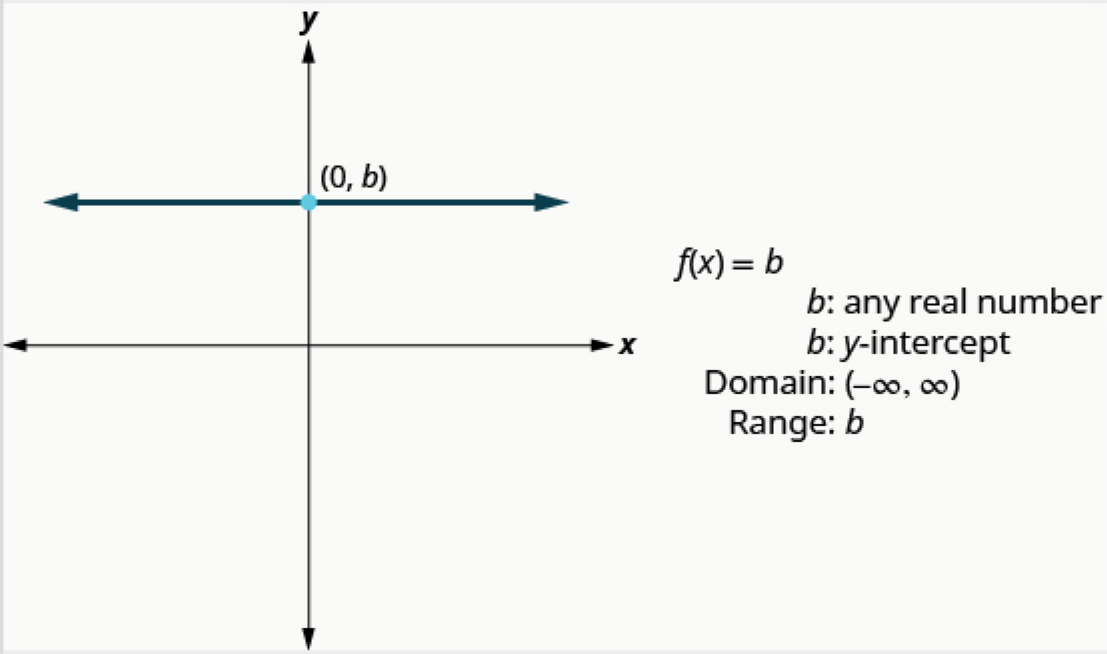
Solution:



The next function whose graph we will look at is called the constant function and its equation is of the form $f\left(x\right)=b$, where b is any real number. If we replace the $f\left(x\right)$ with y, we get $y=b$. We recognize this as the horizontal line whose *y*-intercept is b. The graph of the function $f\left(x\right)=b$, is also the horizontal line whose *y*-intercept is b.

Notice that for any real number we put in the function, the function value will be b. This tells us the range has only one value, b.

Note:
Constant Function



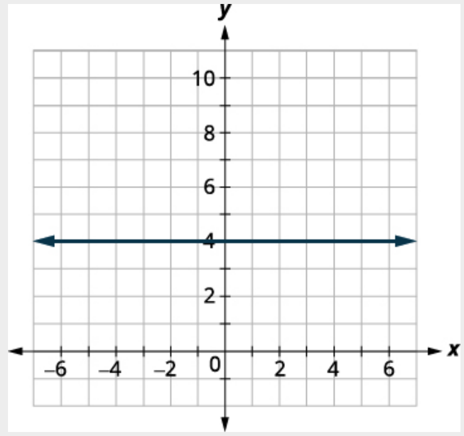
Example:
Exercise:

Problem: Graph: $f\left(x\right)=4$.

Solution:

	$f\left(x\right)=4$
We recognize this as a constant function.	

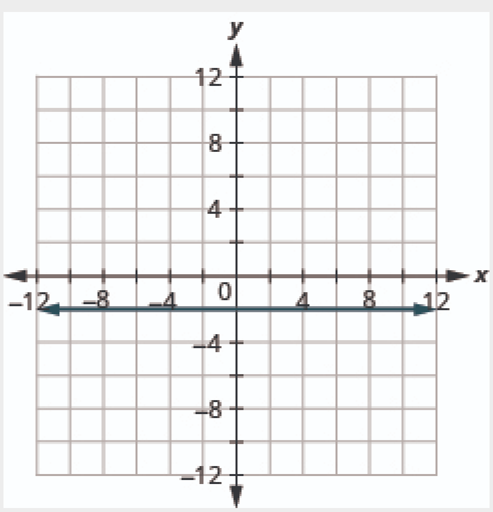
The graph will be a horizontal line through $(0, 4)$.



Note:
Exercise:

Problem: Graph: $f(x) = -2$.

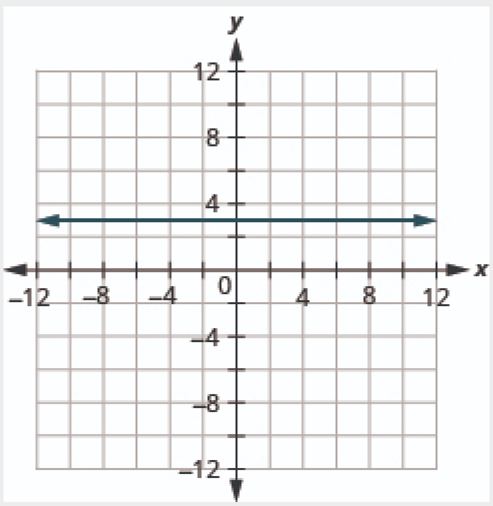
Solution:



Note:
Exercise:

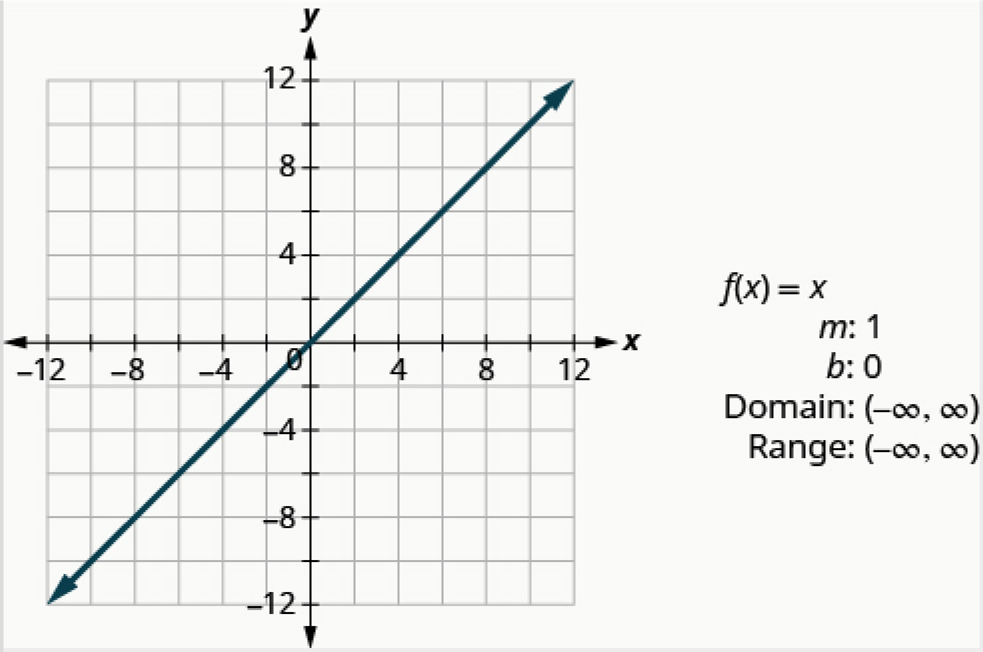
Problem: Graph: $f\left(x\right)=3$.

Solution:



The identity function, $f\left(x\right)=x$ is a special case of the linear function. If we write it in linear function form, $f\left(x\right)=1x+0$, we see the slope is 1 and the *y*-intercept is 0.

Note:
Identity Function



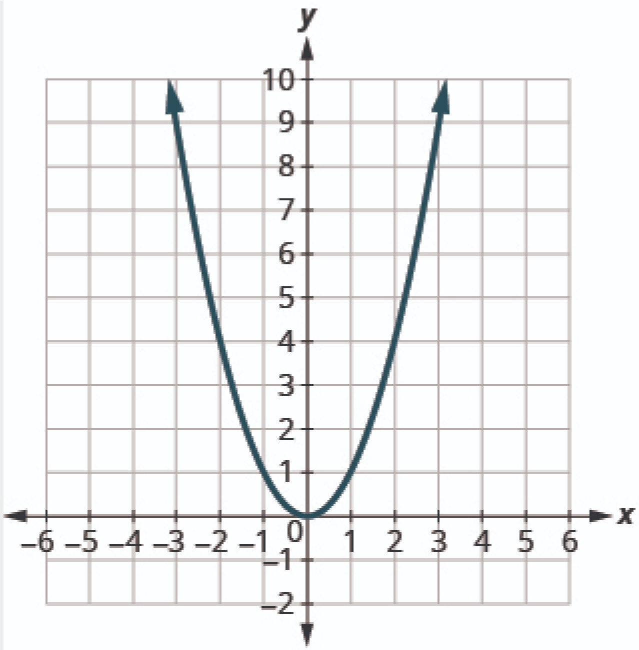
The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

Example:
Exercise:

Problem: Graph: $f\left(x\right)=x^2$.

Solution:

We choose *x*-values. We substitute them in and then create a chart as shown.

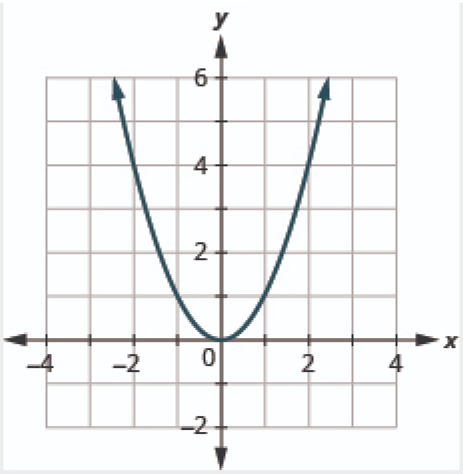


x	$f(x) = x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Note:
Exercise:

Problem: Graph: $f(x) = x^2$.

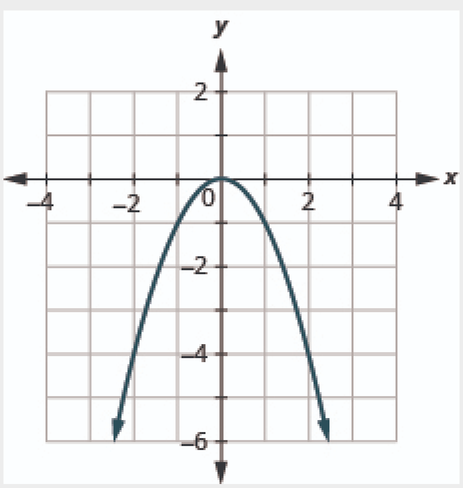
Solution:



Note:
Exercise:

Problem: $f(x) = -x^2$

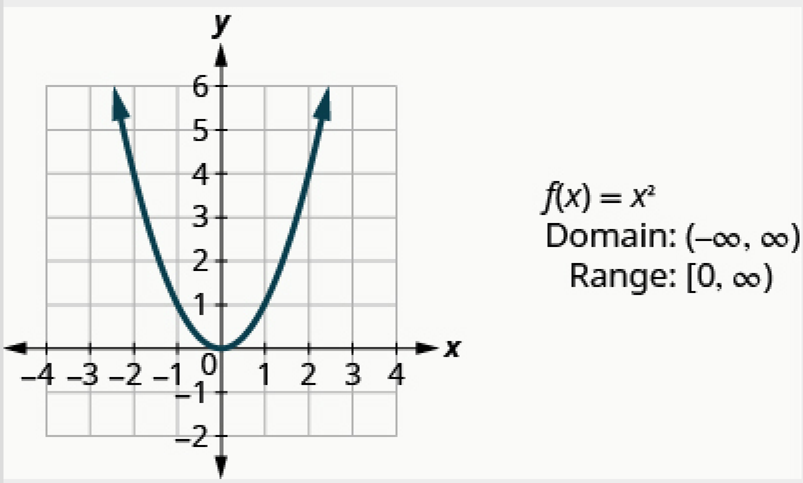
Solution:



Looking at the result in [link], we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of *y* never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

Note:
Square Function



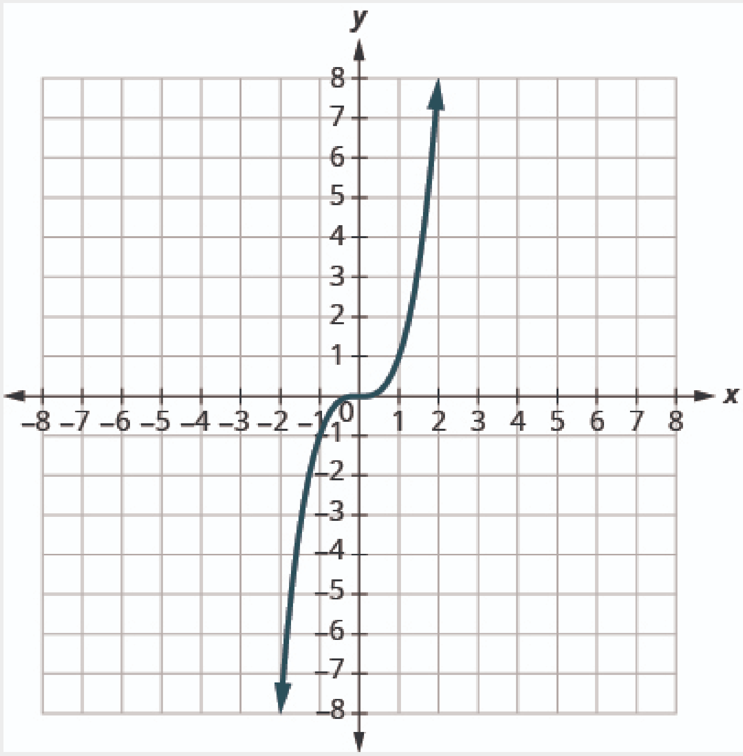
The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

Example:
Exercise:

Problem: Graph: $f\left(x\right)=x^3$.

Solution:

We choose *x*-values. We substitute them in and then create a chart.

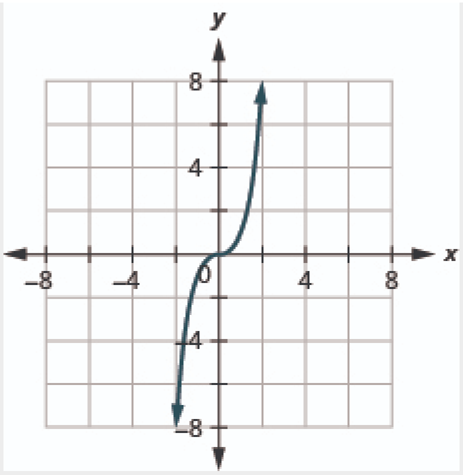


x	$f(x) = x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

Note:
Exercise:

Problem: Graph: $f\left(x\right) = x^3$.

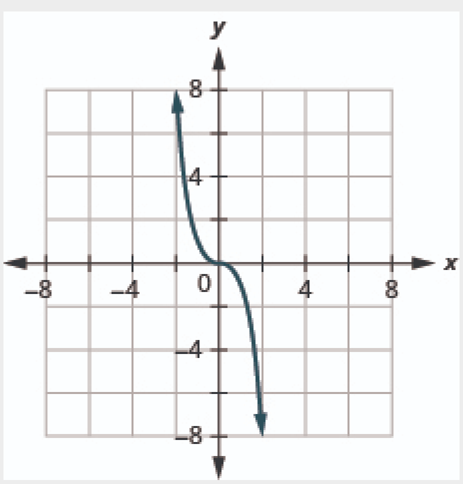
Solution:



Note:
Exercise:

Problem: Graph: $f\left(x\right)=-x^{3}$.

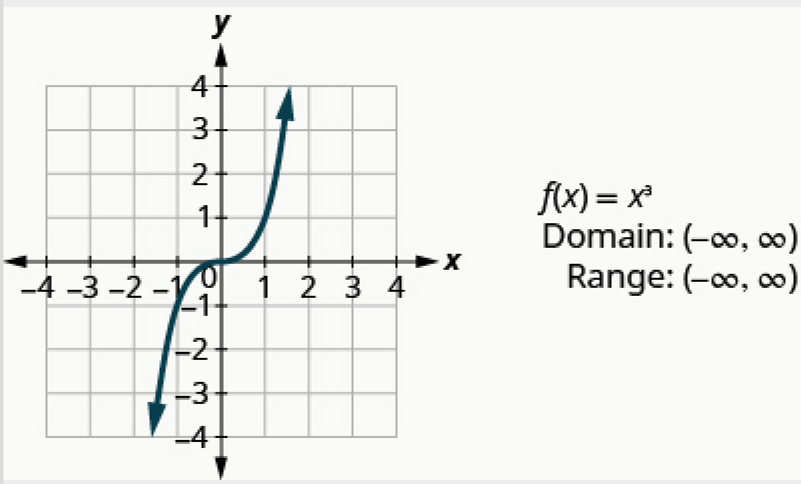
Solution:



Looking at the result in [link], we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

Note:
Cube Function



The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

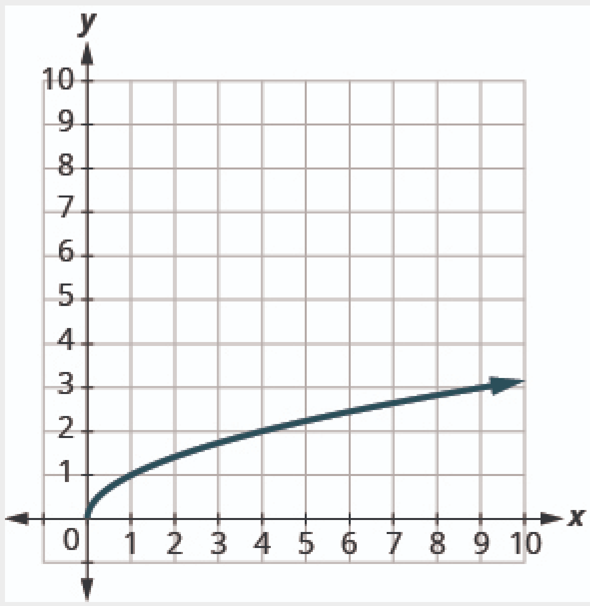
Let’s graph the function $f\left(x\right)=\sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

Example:
Exercise:

Problem: $f\left(x\right)=\sqrt{x}$

Solution:

We choose *x*-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

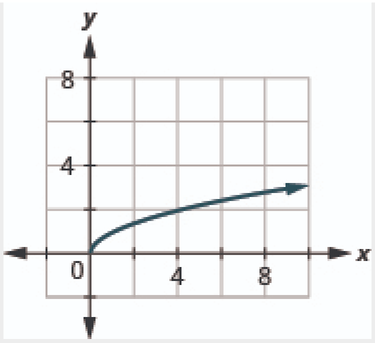


x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Note:
Exercise:

Problem: Graph: $f\left(x\right)=\sqrt{x}$.

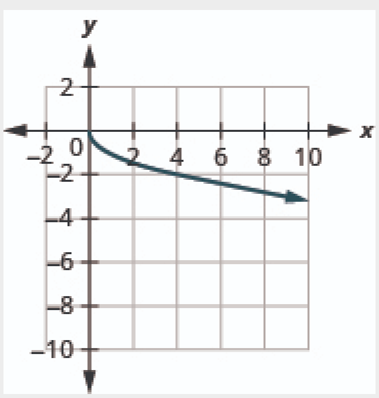
Solution:



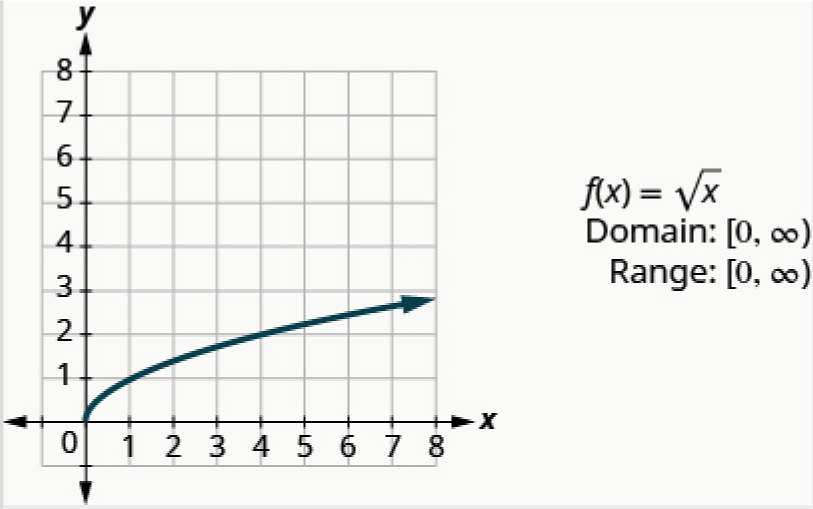
Note:
Exercise:

Problem: Graph: $f\left(x\right)=-\sqrt{x}$.

Solution:



Note:
Square Root Function



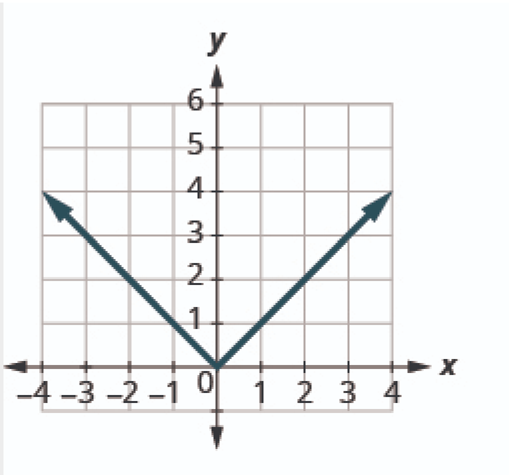
Our last basic function is the absolute value function, $f(x) = |x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

Example:
Exercise:

Problem: Graph: $f(x) = |x|$.

Solution:

We choose *x*-values. We substitute them in and then create a chart.

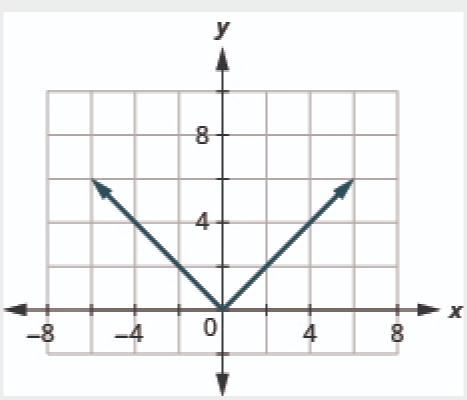


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
–3	3	(–3, 3)
–2	2	(–2, 2)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Note:
Exercise:

Problem: Graph: $f(x) = |x|$.

Solution:

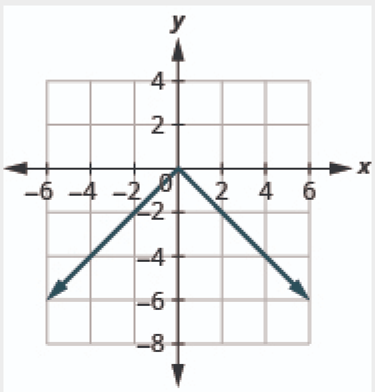


Note:

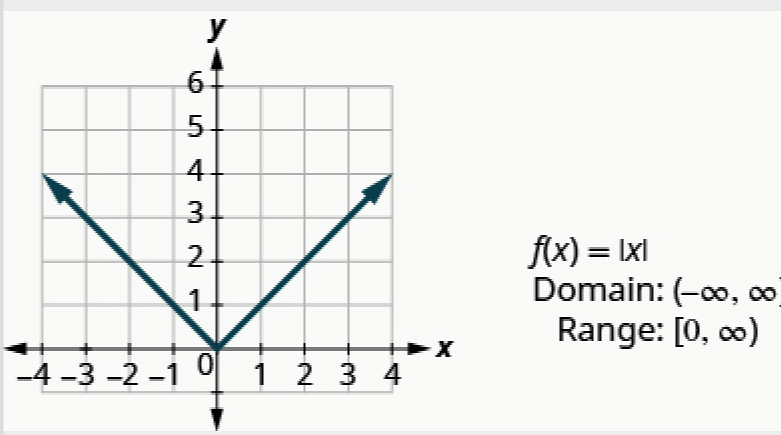
Exercise:

Problem: Graph: $f\left(x\right)=-\left|x\right|.$

Solution:



Note:
Absolute Value Function



Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

We will start by reading the domain and range of a function from its graph.

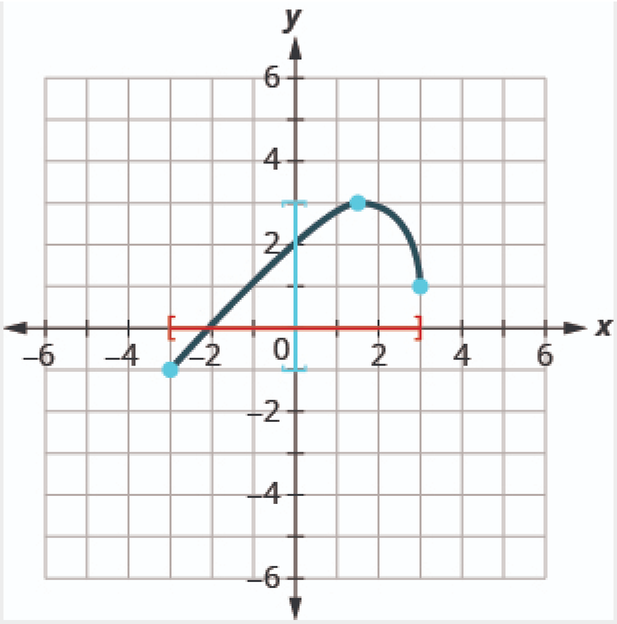
Remember the domain is the set of all the x-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of x that have a corresponding value on the graph. Follow the value x up or down vertically. If you hit the graph of the function then x is in the domain.

Remember the range is the set of all the y-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of y that have a corresponding value on the graph. Follow the value y left or right horizontally. If you hit the graph of the function then y is in the range.

Example:
Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



Solution:

To find the domain we look at the graph and find all the values of *x* that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3, 3]$.

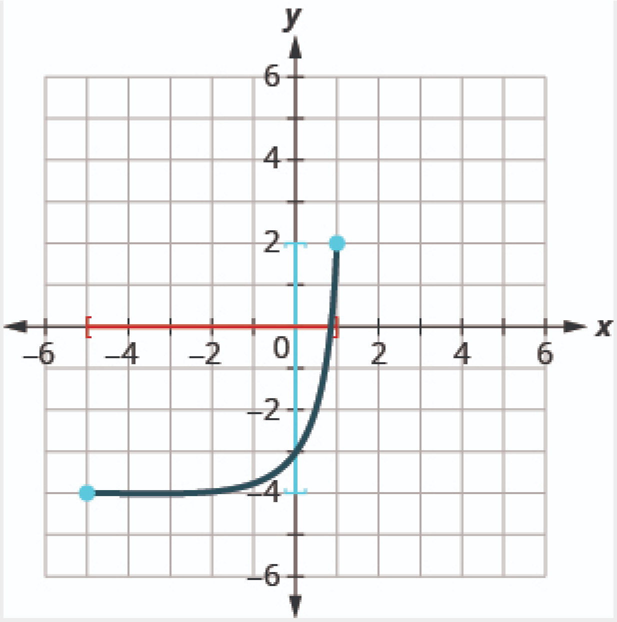
To find the range we look at the graph and find all the values of *y* that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1, 3]$.

Note:

Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



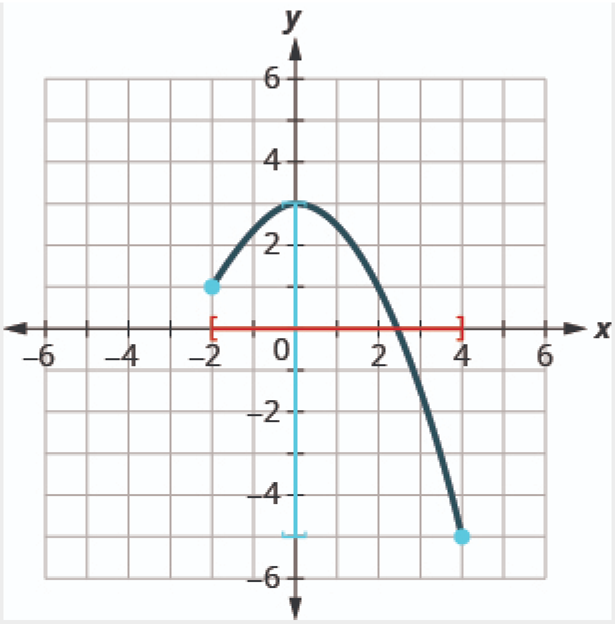
Solution:

The domain is $[-5, 1]$. The range is $[-4, 2]$.

Note:
Exercise:

Problem:

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



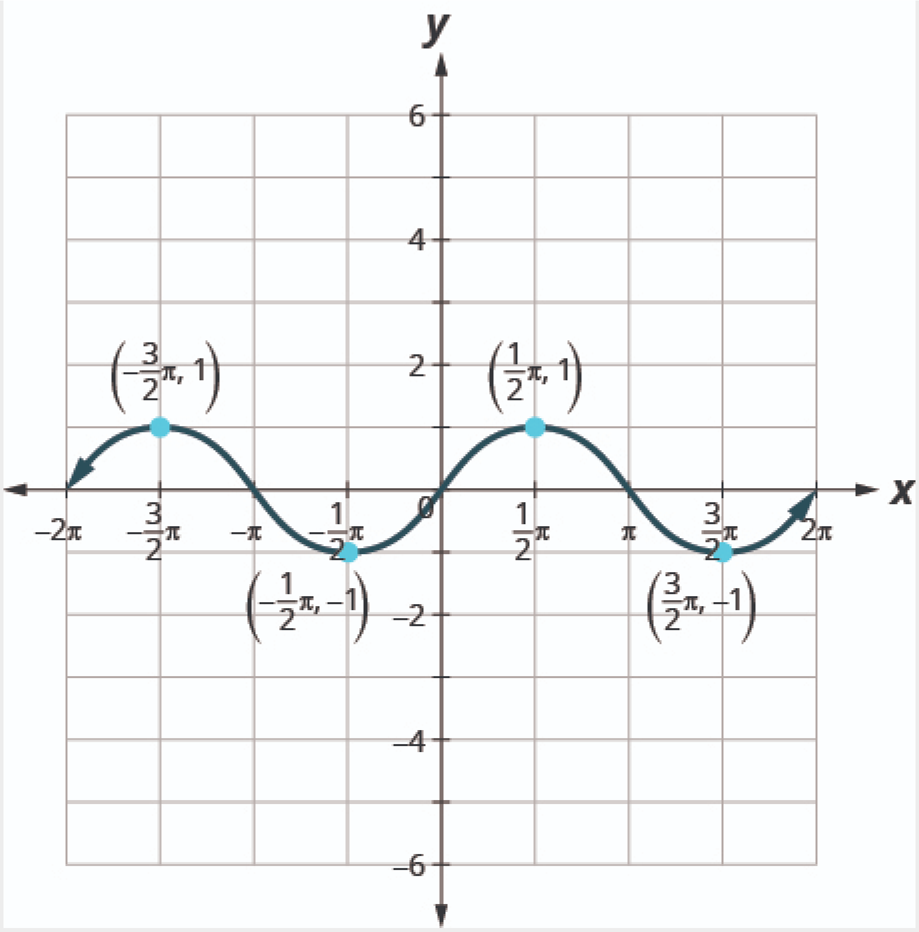
Solution:

The domain is $[-2, 4]$. The range is $[-5, 3]$.

We are now going to read information from the graph that you may see in future math classes.

Example:
Exercise:

Problem: Use the graph of the function to find the indicated values.



ⓐ Find: $f\left(0\right)$.
ⓑ Find: $f\left(\frac{3}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{1}{2}\pi\right)$.
ⓓ Find the values for *x* when $f\left(x\right)=0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

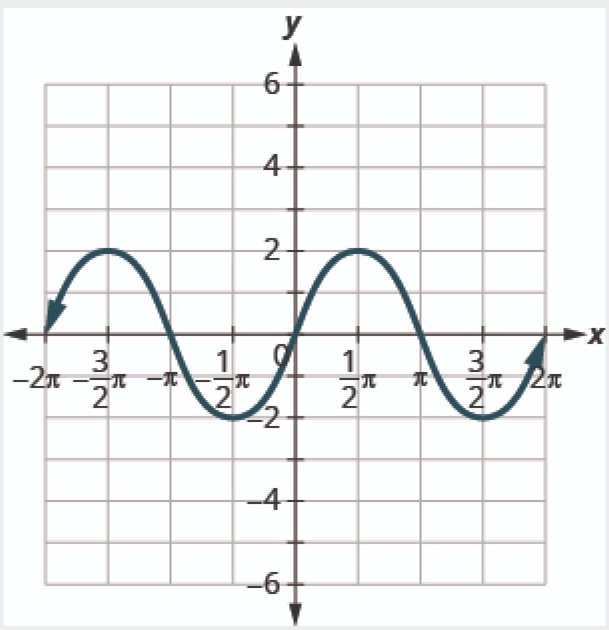
Solution:

ⓐ When $x=0$, the function crosses the *y*-axis at 0. So, $f\left(0\right)=0$.
ⓑ When $x=\frac{3}{2}\pi$, the *y*-value of the function is -1. So, $f\left(\frac{3}{2}\pi\right)=-1$.
ⓒ When $x=-\frac{1}{2}\pi$, the *y*-value of the function is -1. So,

$f\left(-\frac{1}{2}\pi\right) = -1.$
ⓓ The function is 0 at the points, $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$. The *x*-values when $f(x) = 0$ are $-2\pi, -\pi, 0, \pi, 2\pi$.
ⓔ The *x*-intercepts occur when $y = 0$. So the *x*-intercepts occur when $f(x) = 0$. The *x*-intercepts are $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$.
ⓕ The *y*-intercepts occur when $x = 0$. So the *y*-intercepts occur at $f(0)$. The *y*-intercept is $(0, 0)$.
ⓖ This function has a value when *x* is from -2π to 2π. Therefore, the domain in interval notation is $[-2\pi, 2\pi]$.
ⓗ This function values, or *y*-values go from -1 to 1. Therefore, the range, in interval notation, is $[-1, 1]$.

Note:
Exercise:

Problem: Use the graph of the function to find the indicated values.



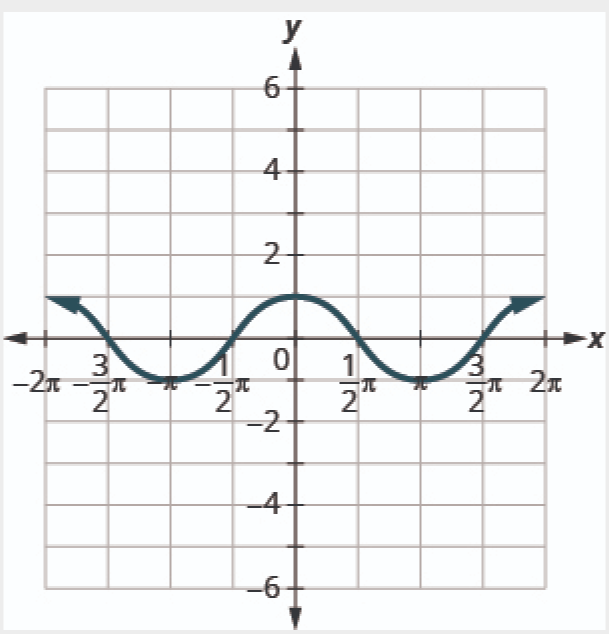
ⓐ Find: $f(0)$.
ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 0$ ⓑ $f = \left(\frac{\pi}{2}\right) = 2$ ⓒ $f = \left(\frac{-3\pi}{2}\right) = 2$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$ ⓔ $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$ ⓕ $(0, 0)$ ⓖ $[-2\pi, 2\pi]$ ⓗ $[-2, 2]$

Note:
Exercise:

Problem: Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 1$ ⓑ $f(\pi) = -1$ ⓒ $f(-\pi) = -1$ ⓓ $f(x) = 0$ for $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ ⓔ $(-2\text{pi}, 0), (-\text{pi}, 0), (0, 0), (\text{pi}, 0), (2\text{pi}, 0)$ ⓕ $(0, 1)$ ⓖ $[-2\text{pi}, 2\text{pi}]$ ⓗ $[-1, 1]$

Note:
Access this online resource for additional instruction and practice with graphs of functions.

- Find Domain and Range

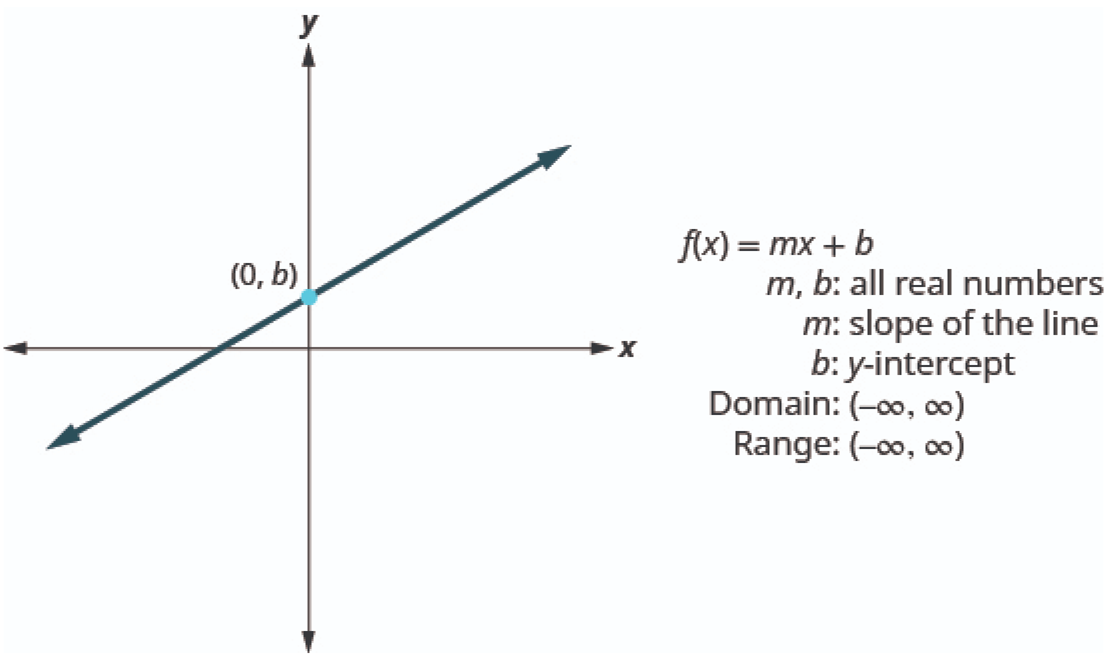
Key Concepts

- **Vertical Line Test**
  - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
  - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
- **Graph of a Function**

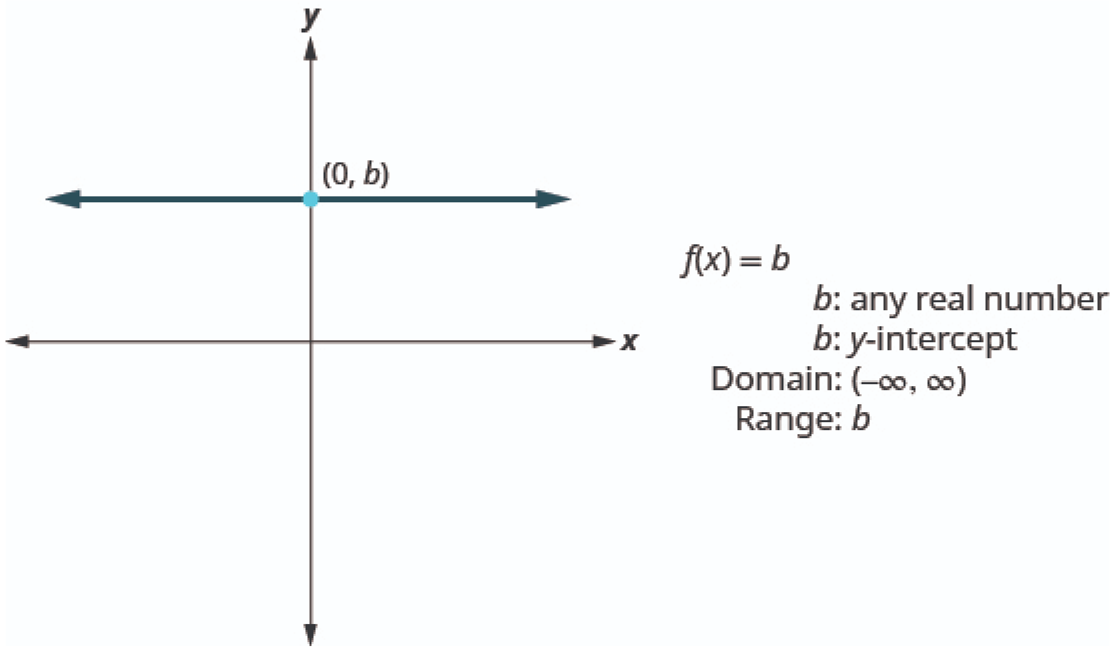
- The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.
  Equation:

$$\begin{array}{rl} f & \text{name of function} \\ x & x\text{-coordinate of the ordered pair} \\ f(x) & y\text{-coordinate of the ordered pair} \end{array}$$

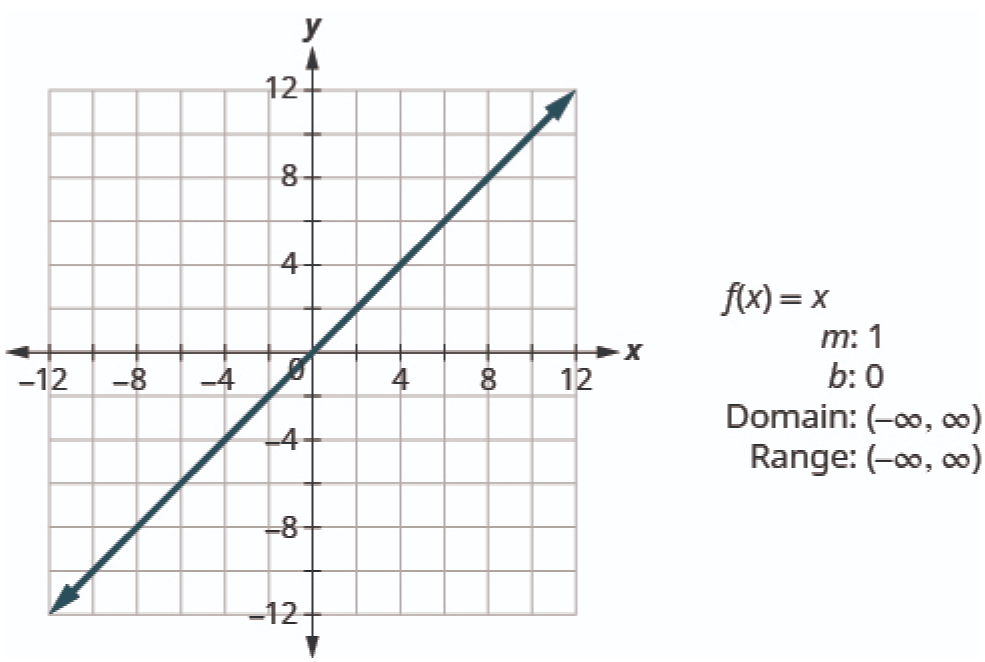
- **Linear Function**



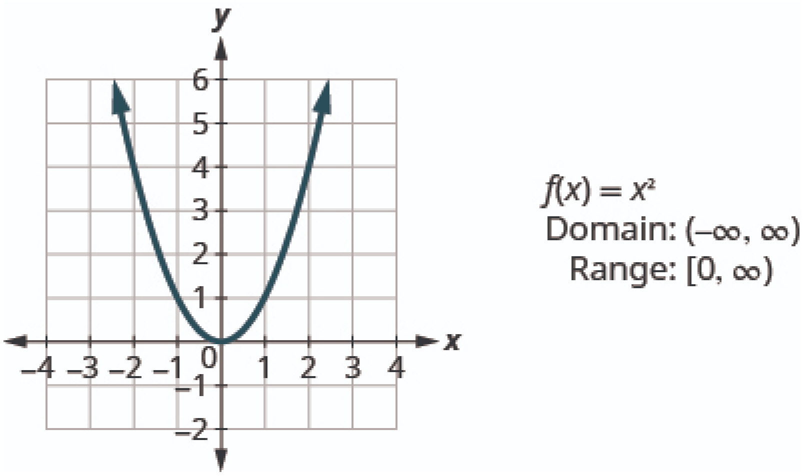
- **Constant Function**



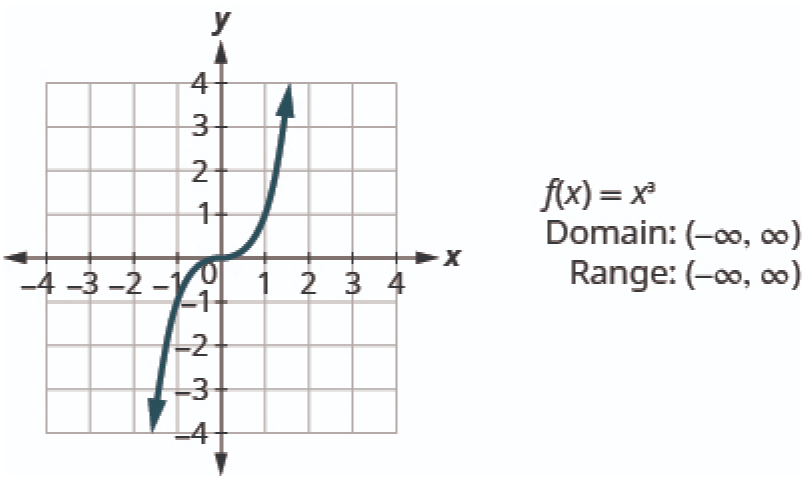
- **Identity Function**



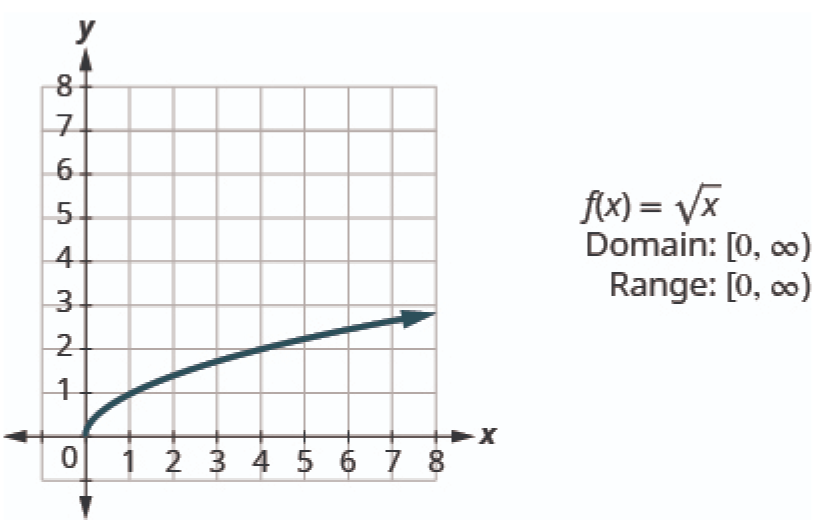
- **Square Function**



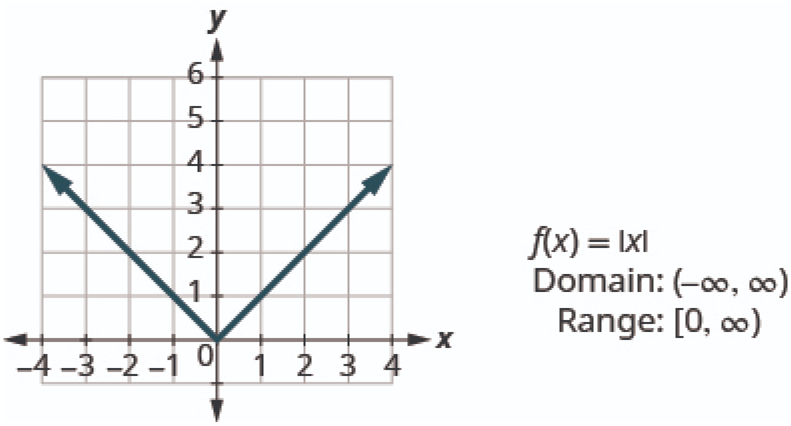
- **Cube Function**



- **Square Root Function**



- **Absolute Value Function**



Section Exercises

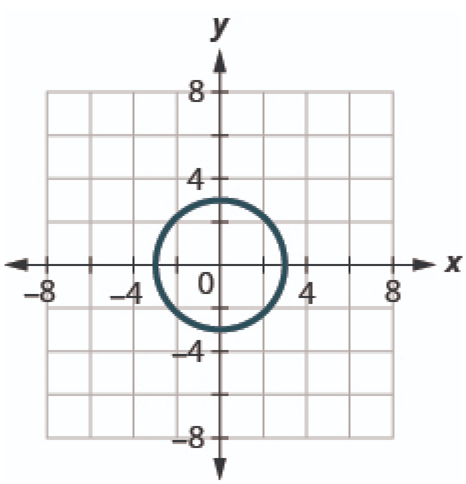
Practice Makes Perfect

Use the Vertical Line Test

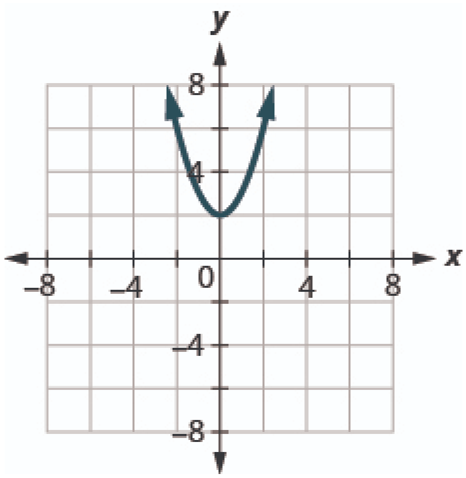
In the following exercises, determine whether each graph is the graph of a function.

Exercise:

Problem: ⓐ



ⓑ

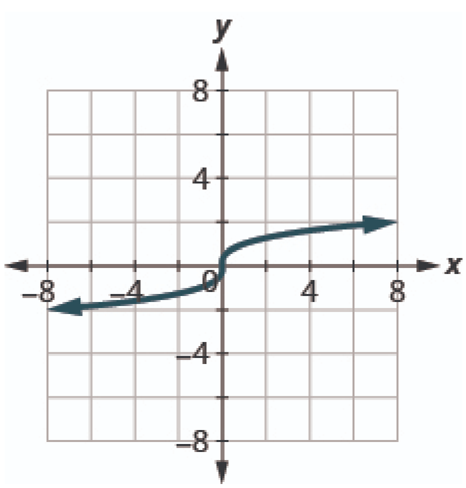


Solution:

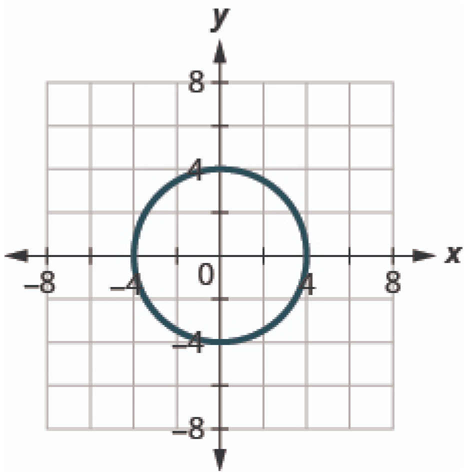
ⓐ no ⓑ yes

Exercise:

Problem: ⓐ

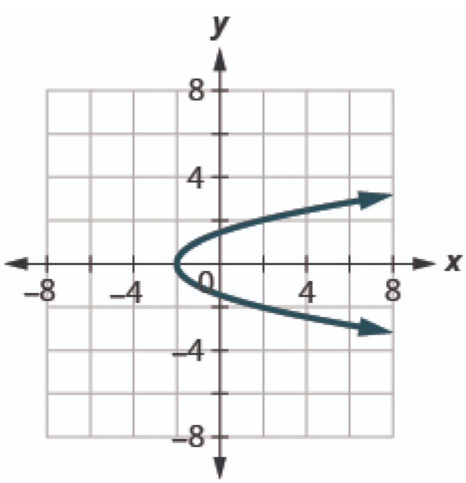


ⓑ

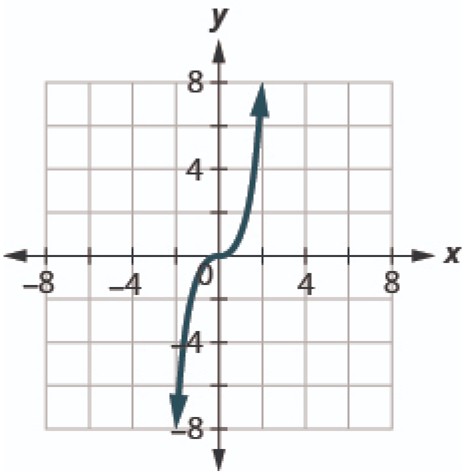


Exercise:

Problem: ⓐ



ⓑ

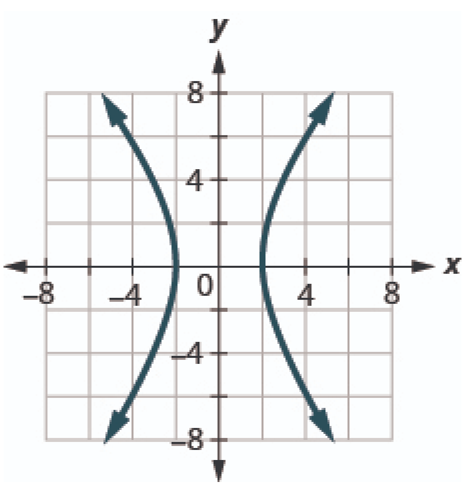


Solution:

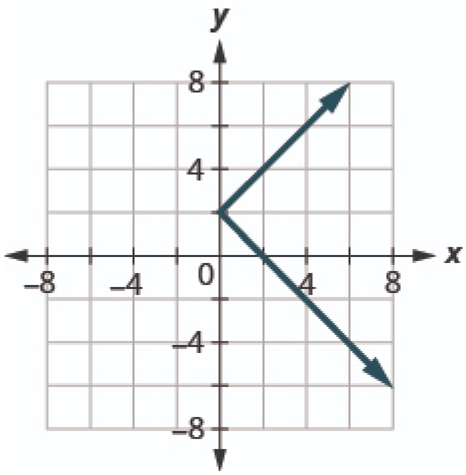
ⓐ no ⓑ yes

Exercise:

Problem: ⓐ



ⓑ



Identify Graphs of Basic Functions

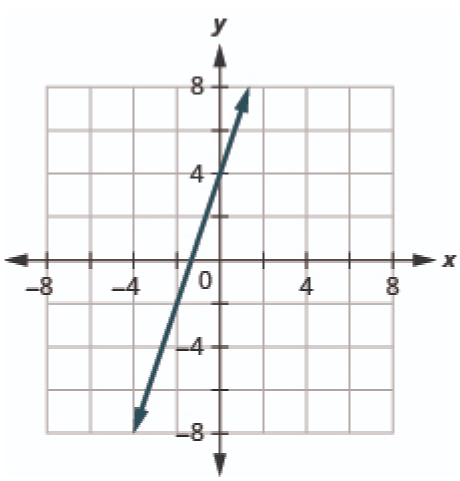
In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

Exercise:

Problem: $f\left(x\right)=3x+4$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

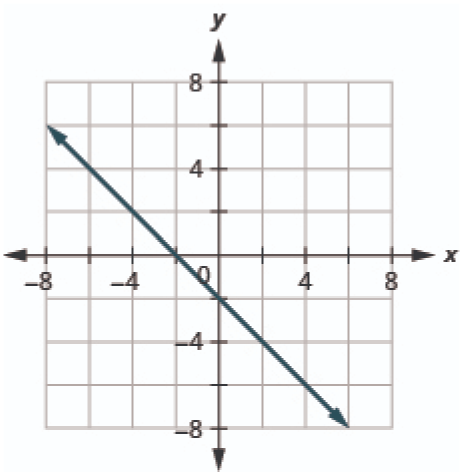
Problem: $f\left(x\right)=2x+5$

Exercise:

Problem: $f\left(x\right)=-x-2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

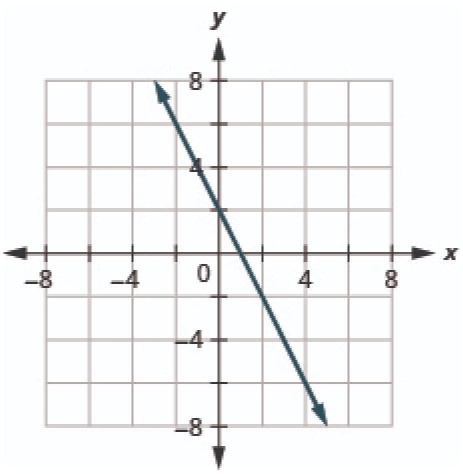
Problem: $f\left(x\right)=-4x-3$

Exercise:

Problem: $f\left(x\right)=-2x+2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

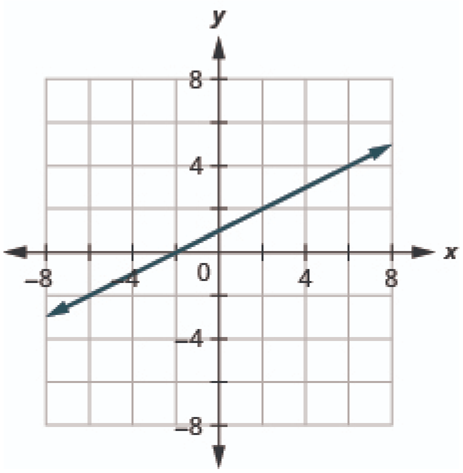
Problem: $f\left(x\right)=-3x+3$

Exercise:

Problem: $f\left(x\right)=\frac{1}{2}x+1$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

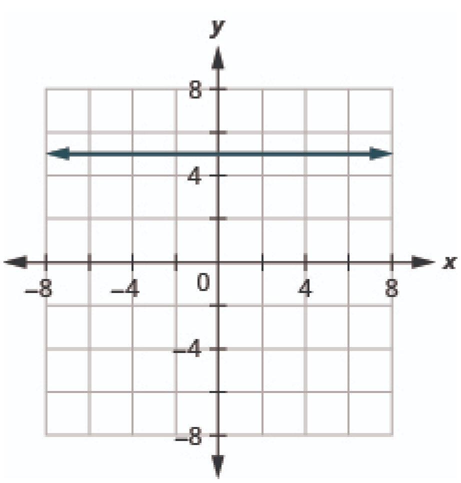
Problem: $f(x) = \frac{2}{3}x - 2$

Exercise:

Problem: $f(x) = 5$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:{5}

Exercise:

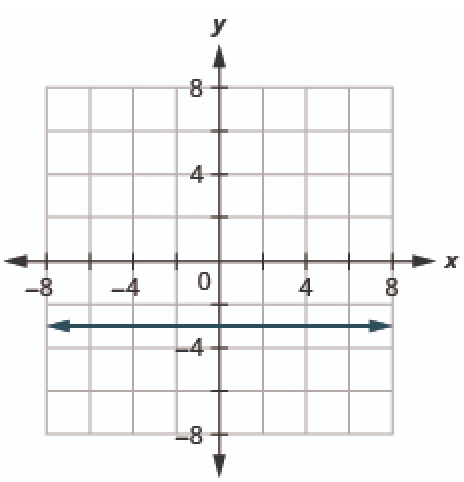
Problem: $f\left(x\right)=2$

Exercise:

Problem: $f\left(x\right)=-3$

Solution:

ⓐ



ⓑ D:(-∞,∞), R: $\{-3\}$

Exercise:

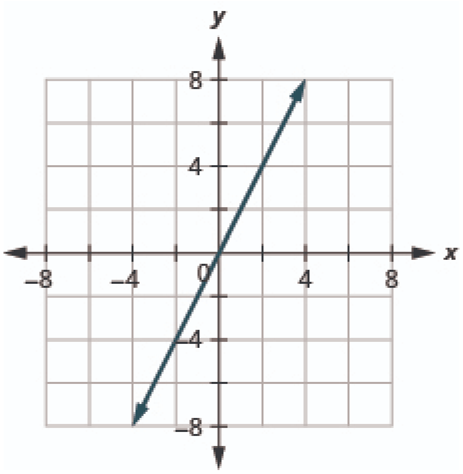
Problem: $f\left(x\right)=-1$

Exercise:

Problem: $f\left(x\right)=2x$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

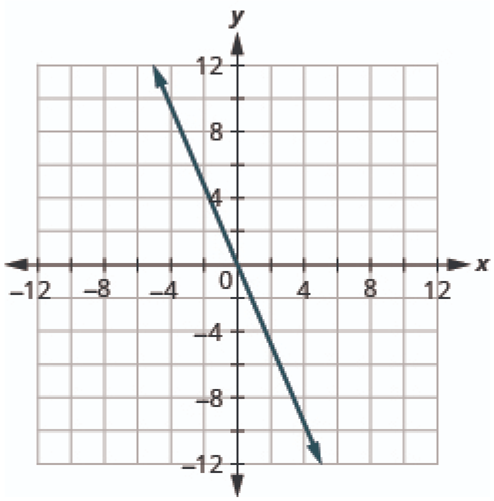
Problem: $f\left(x\right)=3x$

Exercise:

Problem: $f\left(x\right)=-2x$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

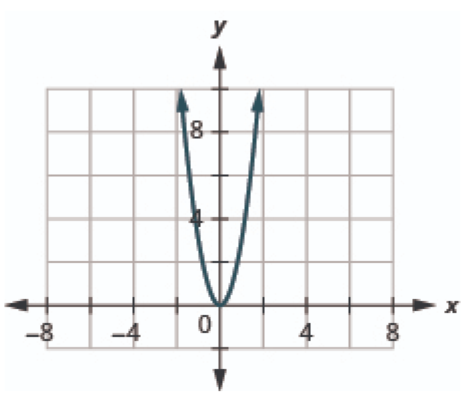
Problem: $f\left(x\right)=-3x$

Exercise:

Problem: $f\left(x\right)=3{x}^{2}$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:[0,∞)

Exercise:

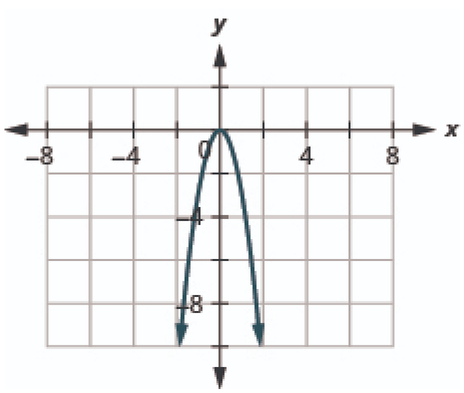
Problem: $f(x) = 2x^2$

Exercise:

Problem: $f(x) = -3x^2$

Solution:

ⓐ



ⓑ (-∞,∞), R:(-∞,0]

Exercise:

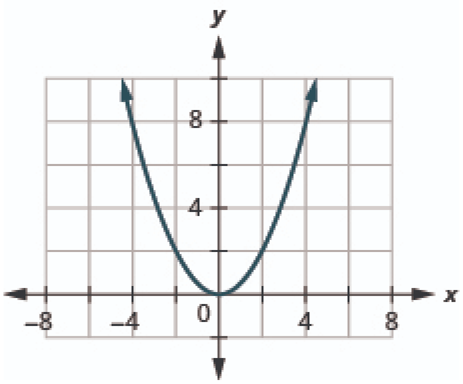
Problem: $f(x) = -2x^2$

Exercise:

Problem: $f(x) = \frac{1}{2}x^2$

Solution:

ⓐ



ⓑ (-∞,∞), R:[-∞,0)

Exercise:

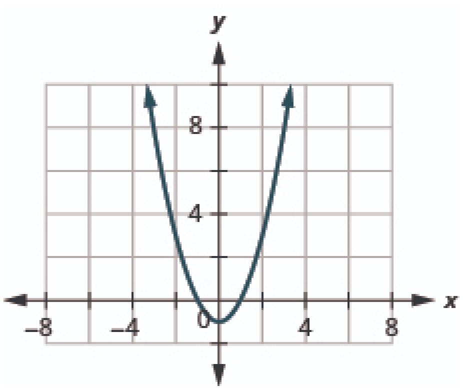
Problem: $f(x) = \frac{1}{3}x^2$

Exercise:

Problem: $f(x) = x^2 - 1$

Solution:

ⓐ



ⓑ (-∞,∞), R:$[-1, \infty)$

Exercise:

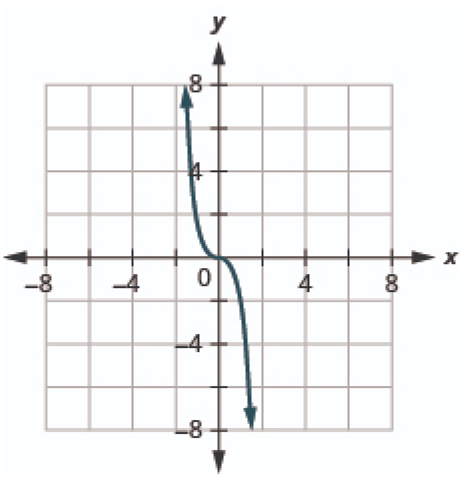
Problem: $f(x) = x^2 + 1$

Exercise:

Problem: $f(x) = -2x^3$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

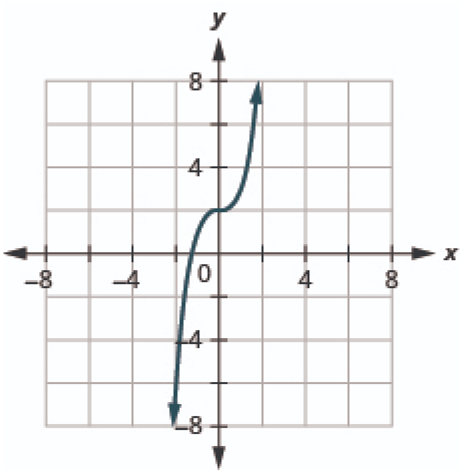
Problem: $f(x) = 2x^3$

Exercise:

Problem: $f(x) = x^3 + 2$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:(-∞,∞)

Exercise:

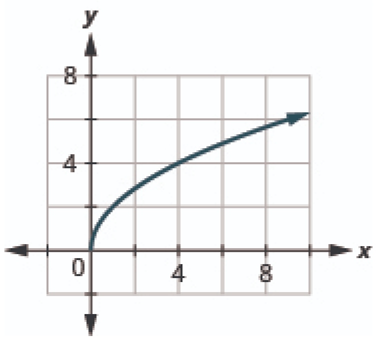
Problem: $f(x) = x^3 - 2$

Exercise:

Problem: $f(x) = 2\sqrt{x}$

Solution:

ⓐ



ⓑ D:[0,∞), R:[0,∞)

Exercise:

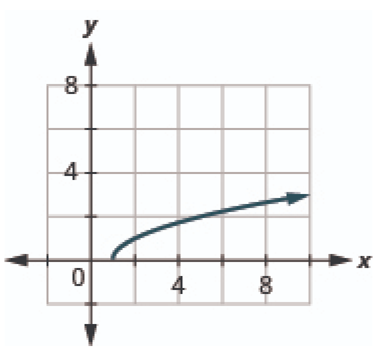
Problem: $f\left(x\right)=-2\sqrt{x}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x-1}$

Solution:

ⓐ



ⓑ D:[1,∞), R:[0,∞)

Exercise:

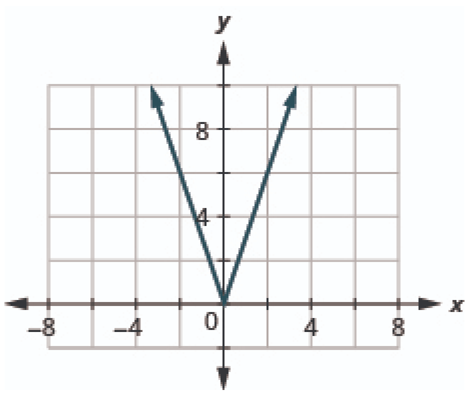
Problem: $f(x) = \sqrt{x+1}$

Exercise:

Problem: $f(x) = 3|x|$

Solution:

ⓐ



ⓑ D:[-1, ∞), R:[−∞,∞)

Exercise:

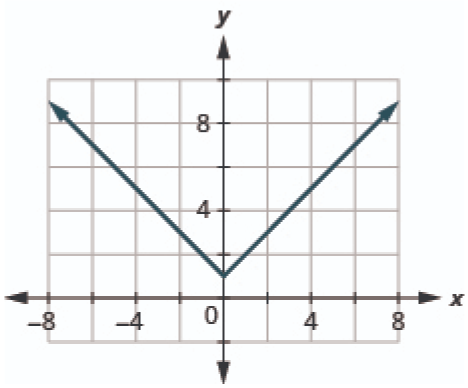
Problem: $f(x) = -2|x|$

Exercise:

Problem: $f(x) = |x| + 1$

Solution:

ⓐ



ⓑ D:(-∞,∞), R:[1,∞)

Exercise:

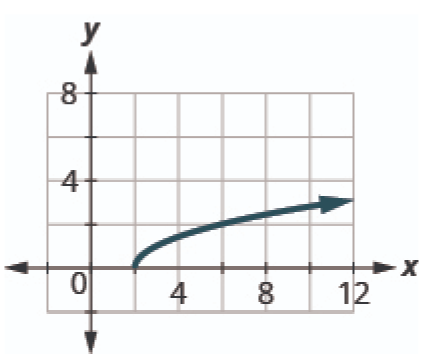
Problem: $f(x) = |x| - 1$

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

Exercise:

Problem:

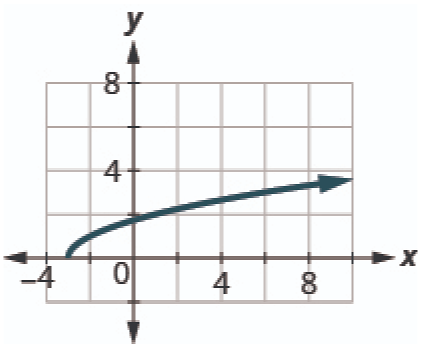


Solution:

D: [2,∞), R: [0,∞)

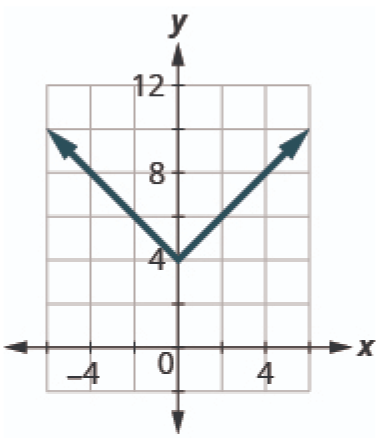
Exercise:

Problem:



Exercise:

Problem:

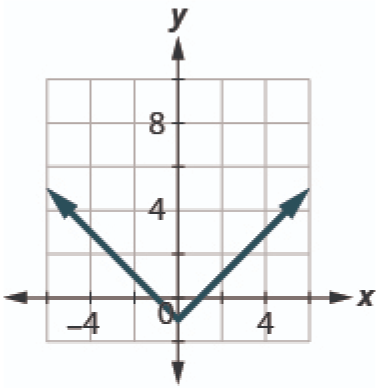


Solution:

D: (-∞,∞), R: [4,∞)

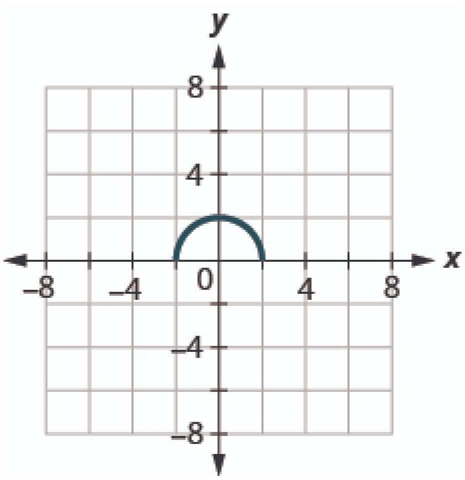
Exercise:

Problem:



Exercise:

Problem:

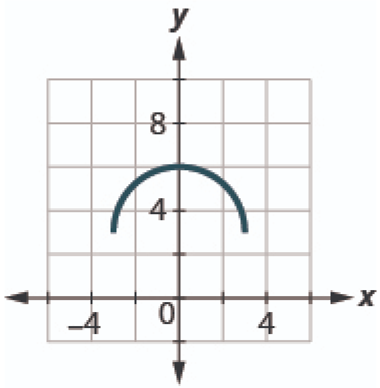


Solution:

D: $[-2, 2]$, R: [0, 2]

Exercise:

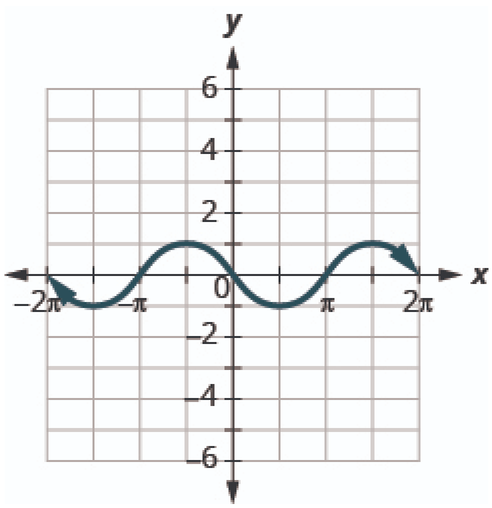
Problem:



In the following exercises, use the graph of the function to find the indicated values.

Exercise:

Problem:



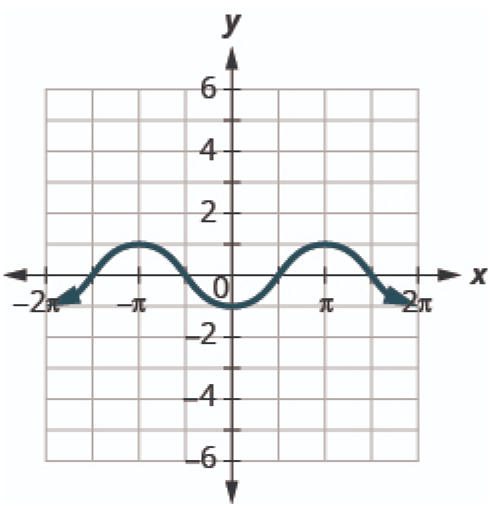
ⓐ Find: $f(0)$.
ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = 0$ ⓑ $(\text{pi}/2) = -1$
ⓒ $f(-3\text{pi}/2) = -1$ ⓓ $f(x) = 0$ for $x = -2\text{pi}, -\text{pi}, 0, \text{pi}, 2\text{pi}$
ⓔ $(-2\text{pi}, 0), (-\text{pi}, 0), (0, 0), (\text{pi}, 0), (2\text{pi}, 0)$ ⓕ $(0, 0)$
ⓖ $[-2\text{pi}, 2\text{pi}]$ ⓗ $[-1, 1]$

Exercise:

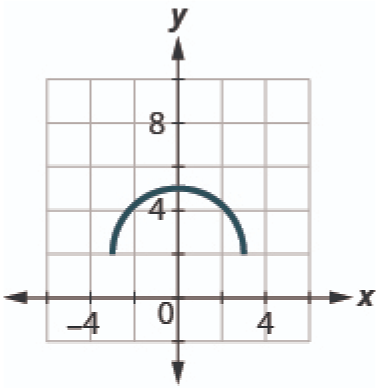
Problem:



ⓐ Find: $f(0)$.
ⓑ Find: $f(\pi)$.
ⓒ Find: $f(-\pi)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation

Exercise:

Problem:



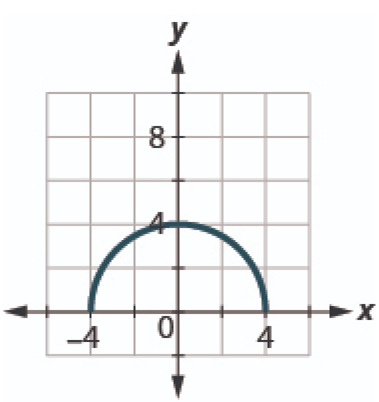
ⓐ Find: $f(0)$.
ⓑ Find: $f(-3)$.
ⓒ Find: $f(3)$.
ⓓ Find the values for *x* when $f(x) = 0$.
ⓔ Find the *x*-intercepts.
ⓕ Find the *y*-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(0) = -6$ ⓑ $f(-3) = 3$ ⓒ $f(3) = 3$ ⓓ $f(x) = 0$ for no *x* ⓔ none ⓕ $y = 6$ ⓖ $[-3, 3]$
ⓗ $[-3, 6]$

Exercise:

Problem:



ⓐ Find: $f(0)$.
ⓑ Find the values for *x* when $f(x) = 0$.
ⓒ Find the *x*-intercepts.
ⓓ Find the *y*-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation

Writing Exercises

Exercise:

Problem:

Explain in your own words how to find the domain from a graph.

Exercise:

Problem:

Explain in your own words how to find the range from a graph.

Exercise:

Problem: Explain in your own words how to use the vertical line test.

Exercise:

Problem:

Draw a sketch of the square and cube functions. What are the similarities and differences in the graphs?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use the vertical line test.			
identify graphs of basic functions.			
read information from a graph.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Chapter Review Exercises

Graph Linear Equations in Two Variables

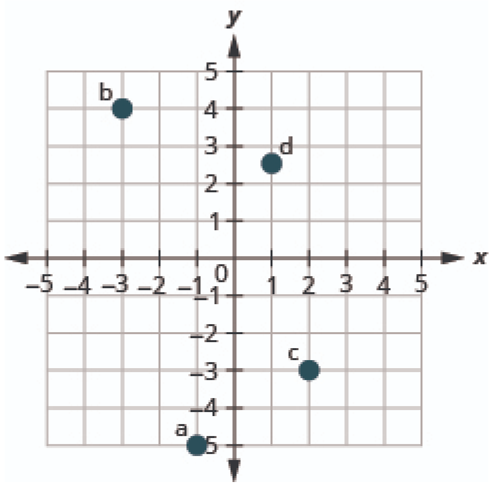
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

Exercise:

Problem:
ⓐ $(-1,-5)$
ⓑ $(-3,4)$
ⓒ $(2,-3)$
ⓓ $\left(1,\frac{5}{2}\right)$

Solution:



Exercise:

Problem: ⓐ $(-2, 0)$
ⓑ $(0, -4)$
ⓒ $(0, 5)$
ⓓ $(3, 0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

Exercise:

Problem: $5x + y = 10;$
ⓐ $(5, 1)$
ⓑ $(2, 0)$
ⓒ $(4, -10)$

Solution:

ⓑ, ⓒ

Exercise:

$y = 6x - 2;$
ⓐ $(1, 4)$
ⓑ $\left(\frac{1}{3}, 0\right)$
Problem: ⓒ $(6, -2)$

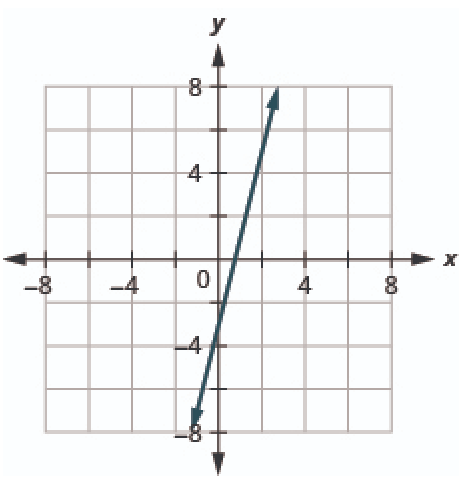
Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

Exercise:

Problem: $y = 4x - 3$

Solution:



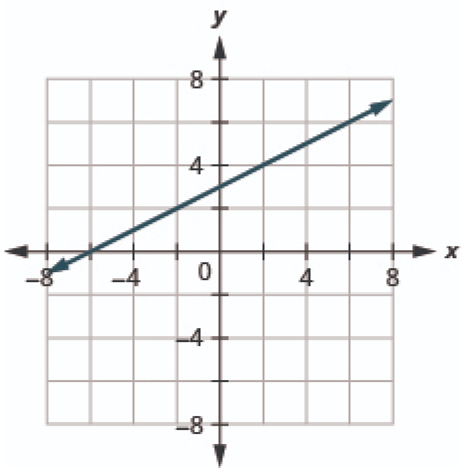
Exercise:

Problem: $y = -3x$

Exercise:

Problem: $y = \frac{1}{2}x + 3$

Solution:



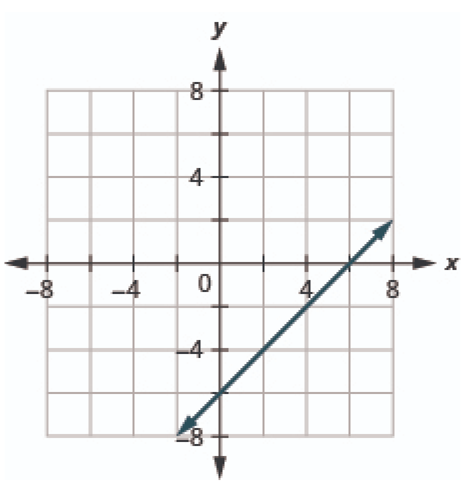
Exercise:

Problem: $y = -\frac{4}{5}x - 1$

Exercise:

Problem: $x - y = 6$

Solution:



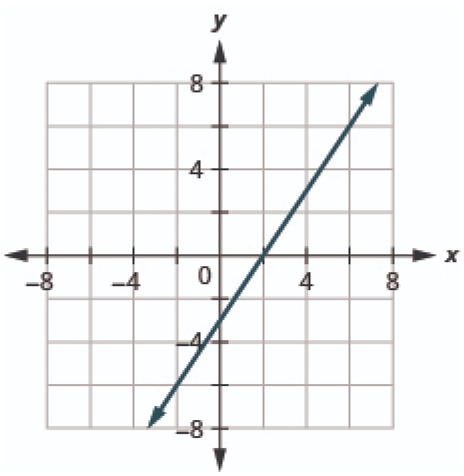
Exercise:

Problem: $2x + y = 7$

Exercise:

Problem: $3x - 2y = 6$

Solution:



Graph Vertical and Horizontal lines

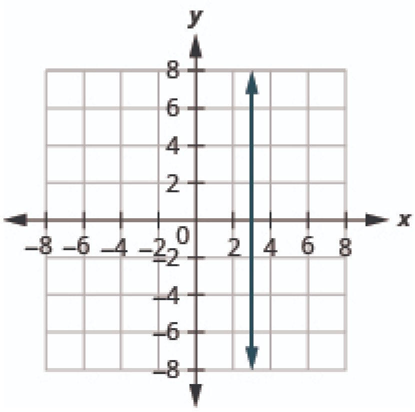
In the following exercises, graph each equation.
Exercise:

Problem: $y = -2$

Exercise:

Problem: $x = 3$

Solution:



In the following exercises, graph each pair of equations in the same rectangular coordinate system.

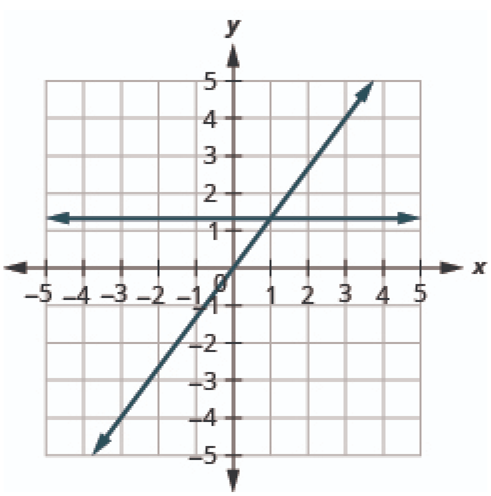
Exercise:

Problem: $y = -2x$ and $y = -2$

Exercise:

Problem: $y = \frac{4}{3}x$ and $y = \frac{4}{3}$

Solution:

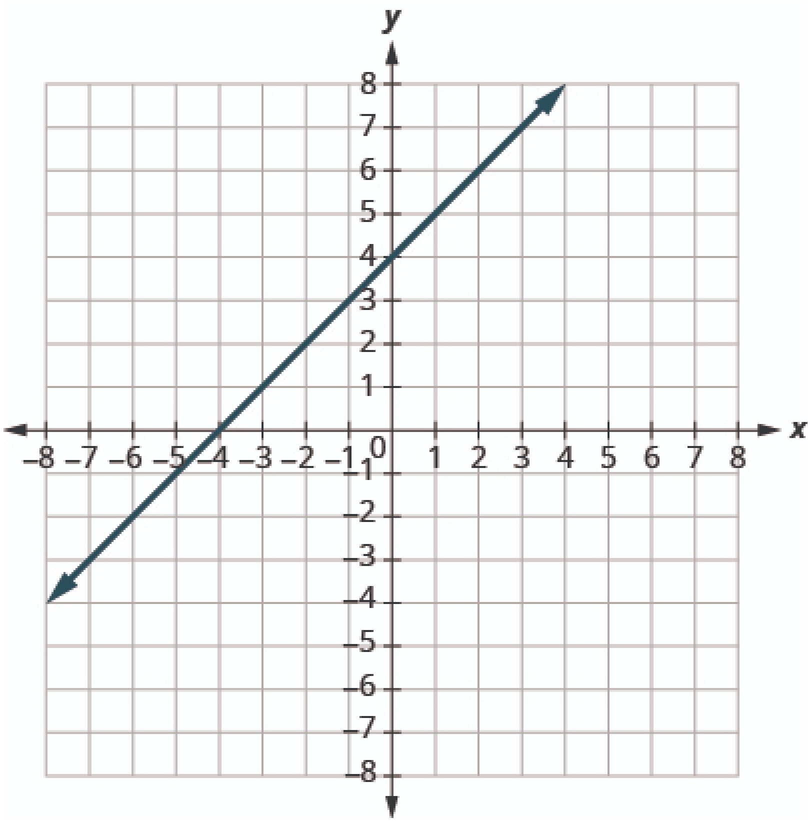


Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts.

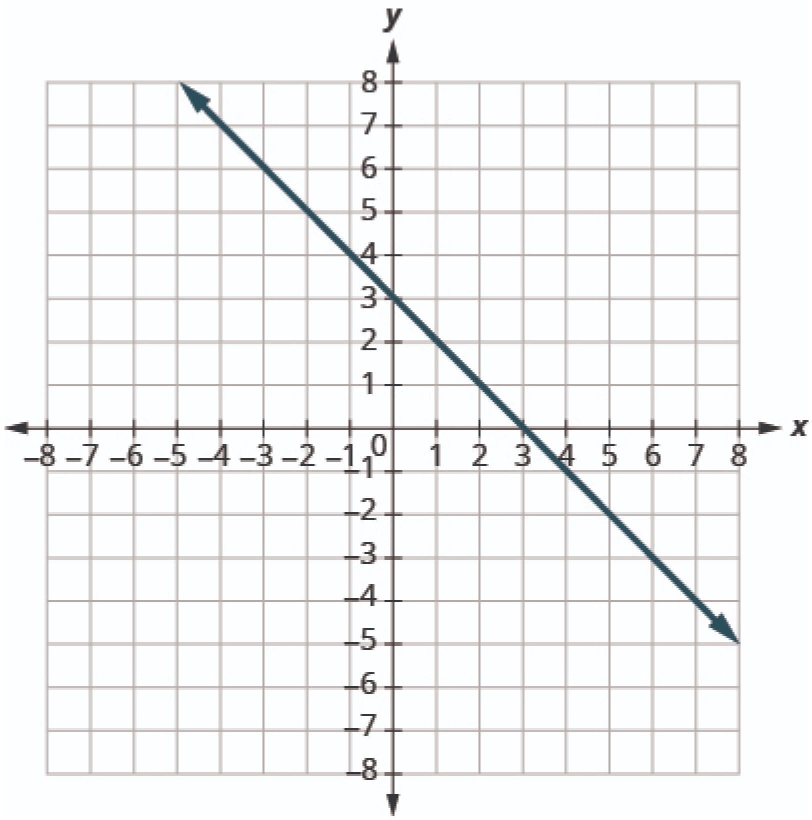
Exercise:

Problem:



Exercise:

Problem:



Solution:

$(0, 3) (3, 0)$

In the following exercises, find the intercepts of each equation.

Exercise:

Problem: $x - y = -1$

Exercise:

Problem: $x + 2y = 6$

Solution:

$(6, 0), (0, 3)$

Exercise:

Problem: $2x + 3y = 12$

Exercise:

Problem: $y = \frac{3}{4}x - 12$

Solution:

$(16, 0), (0, -12)$

Exercise:

Problem: $y = 3x$

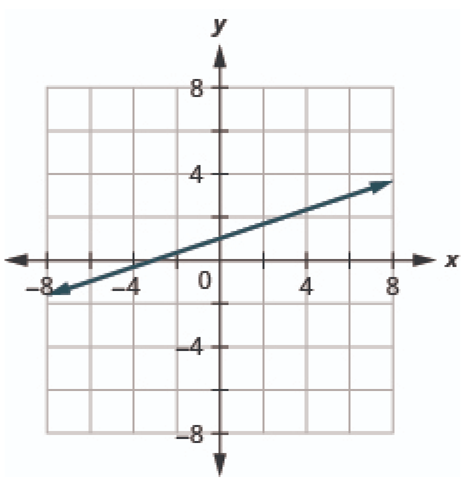
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

Exercise:

Problem: $-x + 3y = 3$

Solution:



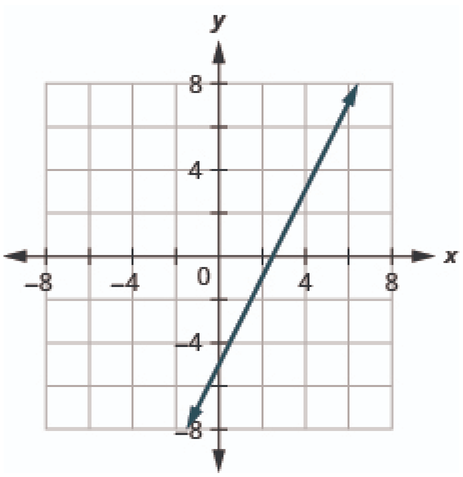
Exercise:

Problem: $x - y = 4$

Exercise:

Problem: $2x - y = 5$

Solution:



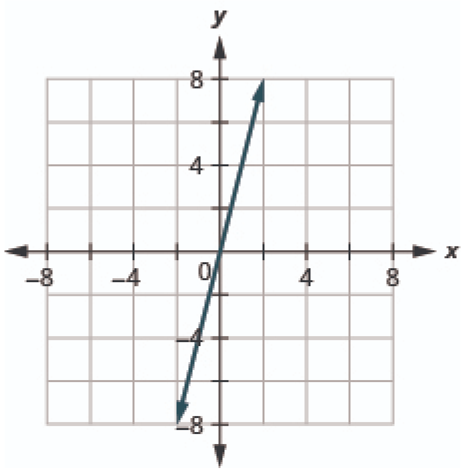
Exercise:

Problem: $2x - 4y = 8$

Exercise:

Problem: $y = 4x$

Solution:



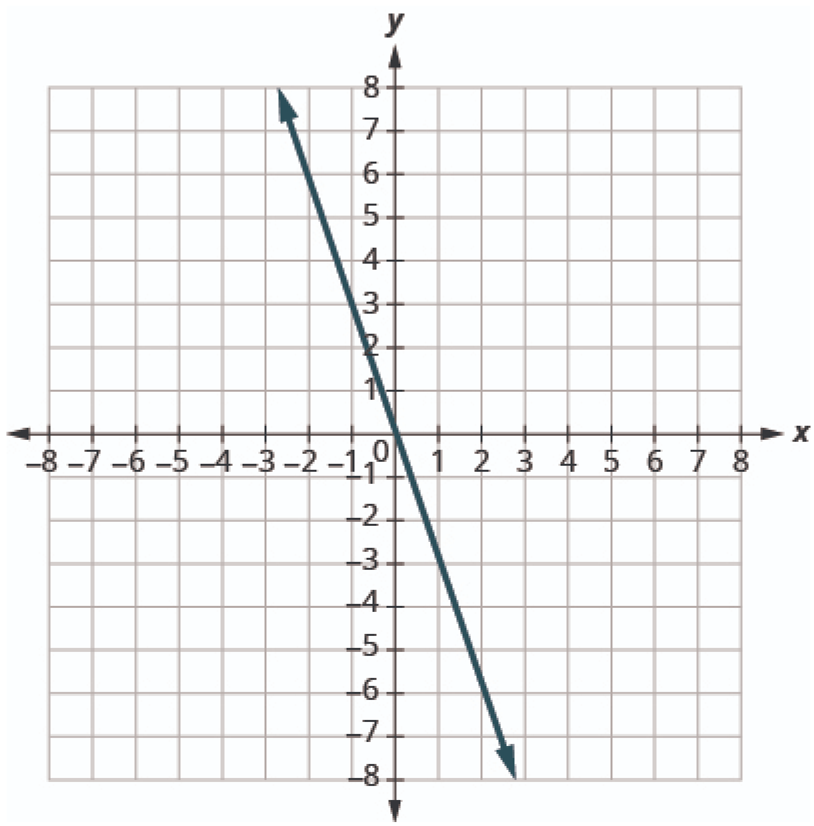
Slope of a Line

Find the Slope of a Line

In the following exercises, find the slope of each line shown.

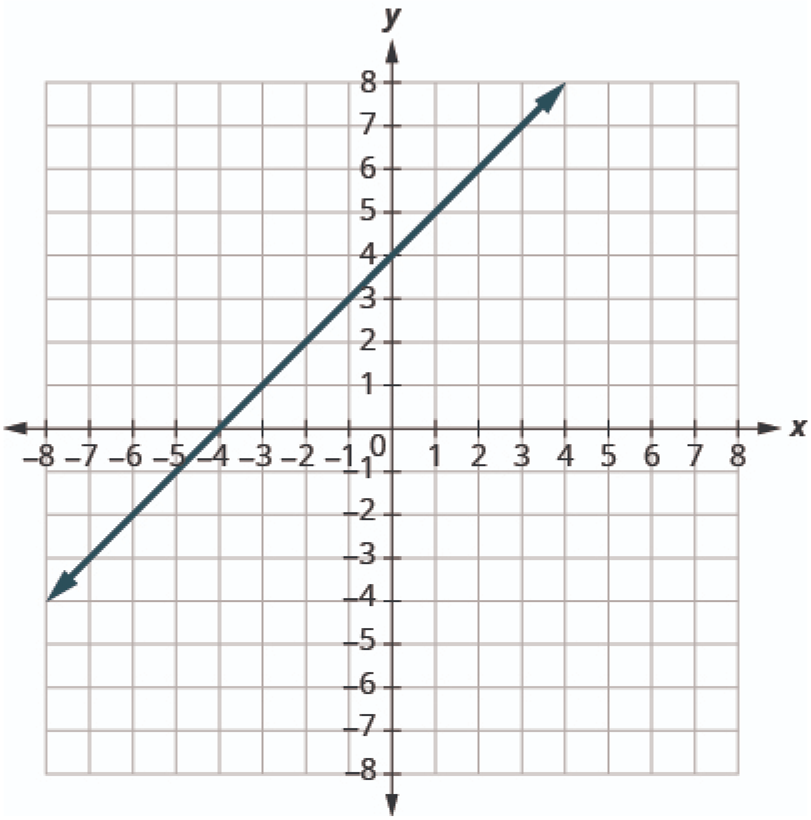
Exercise:

Problem:



Exercise:

Problem:

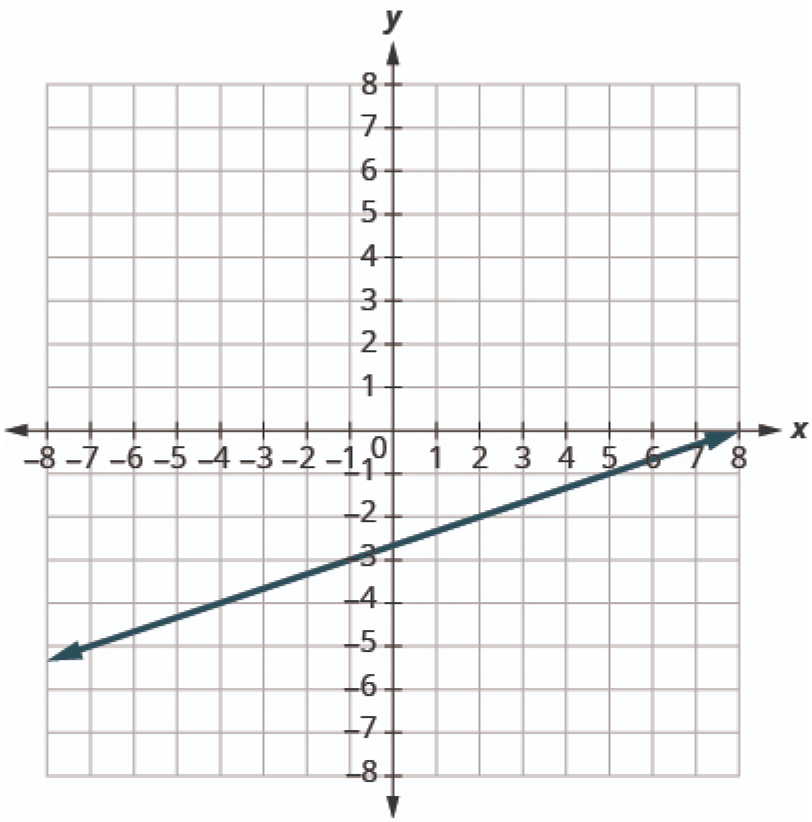


Solution:

1

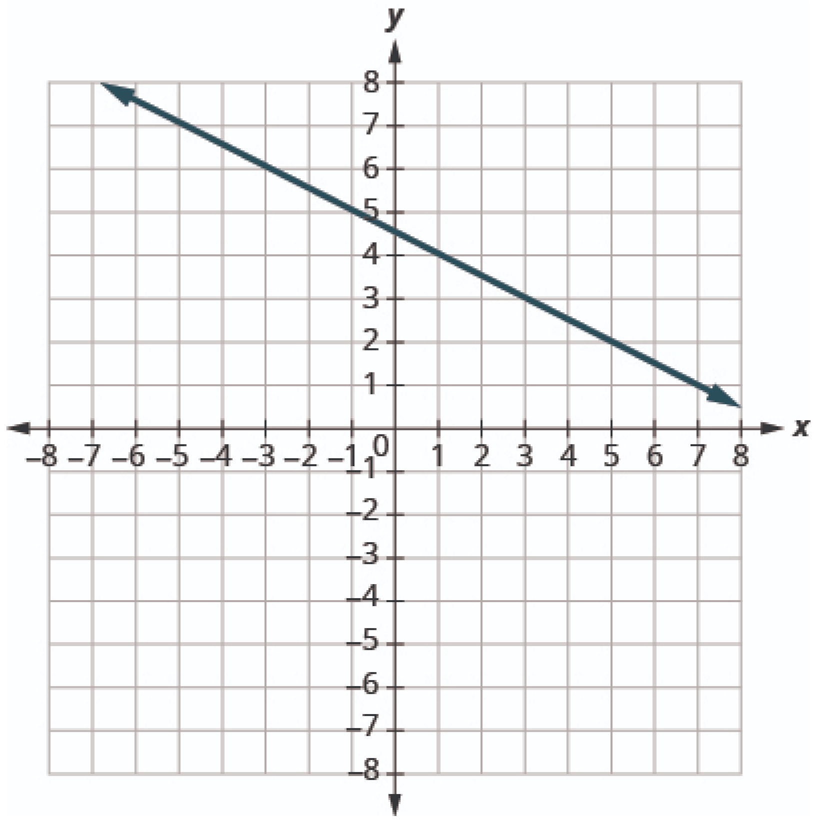
Exercise:

Problem:



Exercise:

Problem:



Solution:

$-\frac{1}{2}$

In the following exercises, find the slope of each line.

Exercise:

Problem: $y = 2$

Exercise:

Problem: $x = 5$

Solution:

undefined

Exercise:

Problem: $x = -3$

Exercise:

Problem: $y = -1$

Solution:

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise:

Problem: $(-1, -1), (0, 5)$

Exercise:

Problem: $(3.5), (4, -1)$

Solution:

-6

Exercise:

Problem: $(-5, -2), (3, 2)$

Exercise:

Problem: $(2, 1), (4, 6)$

Solution:

$\frac{5}{2}$

Graph a Line Given a Point and the Slope

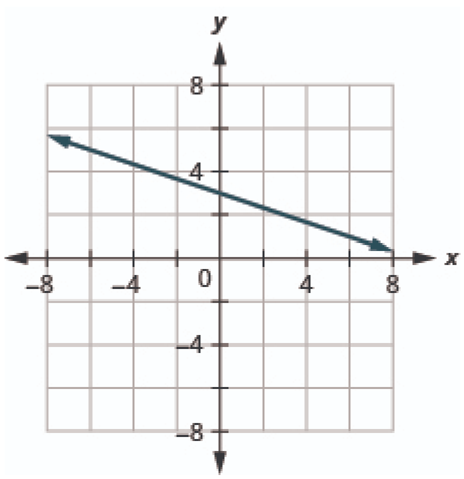
In the following exercises, graph each line with the given point and slope.
Exercise:

Problem: $(2, -2)$; $m = \frac{5}{2}$

Exercise:

Problem: $(-3, 4)$; $m = -\frac{1}{3}$

Solution:



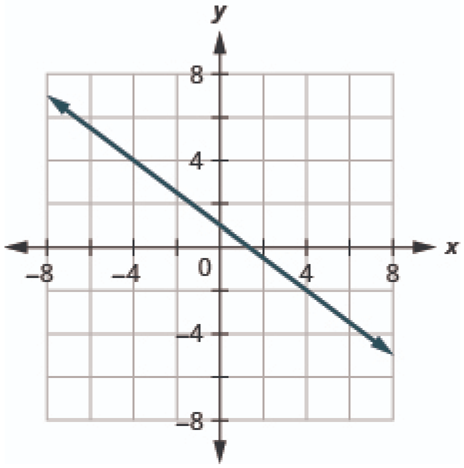
Exercise:

Problem: *x*-intercept -4; $m = 3$

Exercise:

Problem: *y*-intercept 1; $m = -\frac{3}{4}$

Solution:



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and *y*-intercept of each line.

Exercise:

Problem: $y = -4x + 9$

Exercise:

Problem: $y = \frac{5}{3}x - 6$

Solution:

$m = \frac{5}{3}; (0, -6)$

Exercise:

Problem: $5x + y = 10$

Exercise:

Problem: $4x - 5y = 8$

Solution:

$m = \frac{4}{5}$; $\left(0, -\frac{8}{5}\right)$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

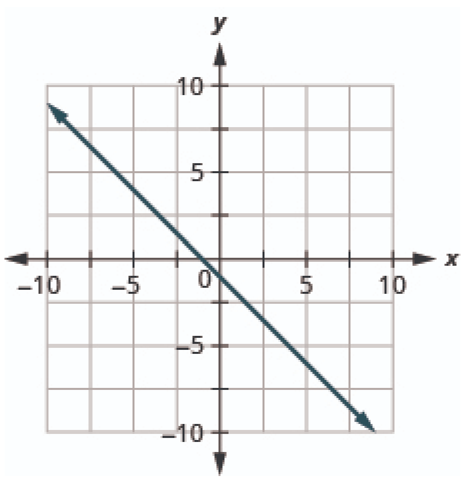
Exercise:

Problem: $y = 2x + 3$

Exercise:

Problem: $y = -x - 1$

Solution:



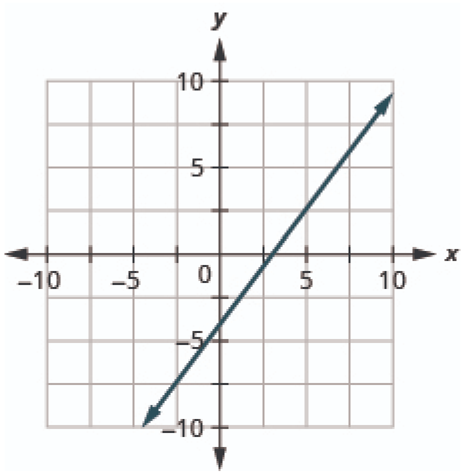
Exercise:

Problem: $y = -\frac{2}{5}x + 3$

Exercise:

Problem: $4x - 3y = 12$

Solution:



In the following exercises, determine the most convenient method to graph each line.

Exercise:

Problem: $x = 5$

Exercise:

Problem: $y = -3$

Solution:

horizontal line

Exercise:

Problem: $2x + y = 5$

Exercise:

Problem: $x - y = 2$

Solution:

intercepts

Exercise:

Problem: $y = \frac{2}{2}x + 2$

Exercise:

Problem: $y = \frac{3}{4}x - 1$

Solution:

plotting points

Graph and Interpret Applications of Slope-Intercept

Exercise:

Problem:

Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, *C*, in dollars and the number of meals, *m*, that she serves.

ⓐ Find Katherine’s cost for a week when she serves no meals.
ⓑ Find the cost for a week when she serves 14 meals.
ⓒ Interpret the slope and *C*-intercept of the equation.
ⓓ Graph the equation.

Exercise:

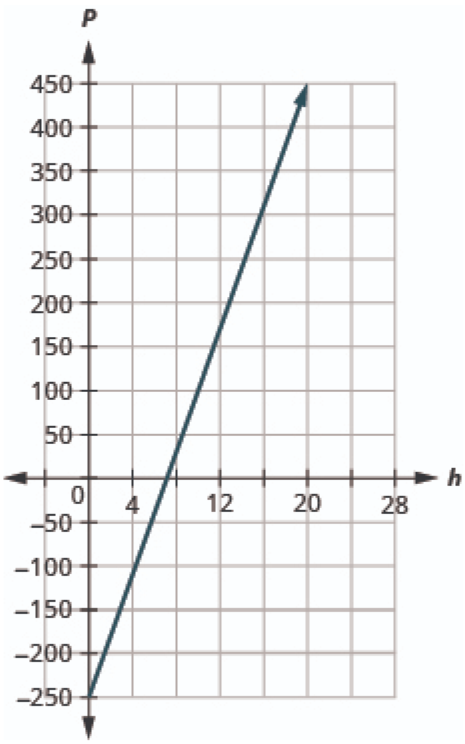
Problem:

Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, *P*, in dollars and the number of student lessons, *s*, that she teaches.

ⓐ Find Marjorie’s profit for a week when she teaches no student lessons.
ⓑ Find the profit for a week when she teaches 20 student lessons.
ⓒ Interpret the slope and *P*-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $-\$250$
ⓑ $\$450$
ⓒ The slope, 35, means that Marjorie’s weekly profit, *P*, increases by $35 for each additional student lesson she teaches.
The *P*-intercept means that when the number of lessons is 0, Marjorie loses $250.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

Exercise:

Problem: $4x - 3y = -1; y = \frac{4}{3}x - 3$

Exercise:

Problem: $y = 5x - 1; 10x + 2y = 0$

Solution:

neither

Exercise:

Problem: $3x - 2y = 5; 2x + 3y = 6$

Exercise:

Problem: $2x - y = 8; x - 2y = 4$

Solution:

not parallel

Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

Exercise:

Problem: slope $\frac{1}{3}$ and y-intercept $(0, -6)$

Exercise:

Problem: slope -5 and y-intercept $(0, -3)$

Solution:

$y = -5x - 3$

Exercise:

Problem: slope 0 and y-intercept $(0,4)$

Exercise:

Problem: slope -2 and y-intercept $(0,0)$

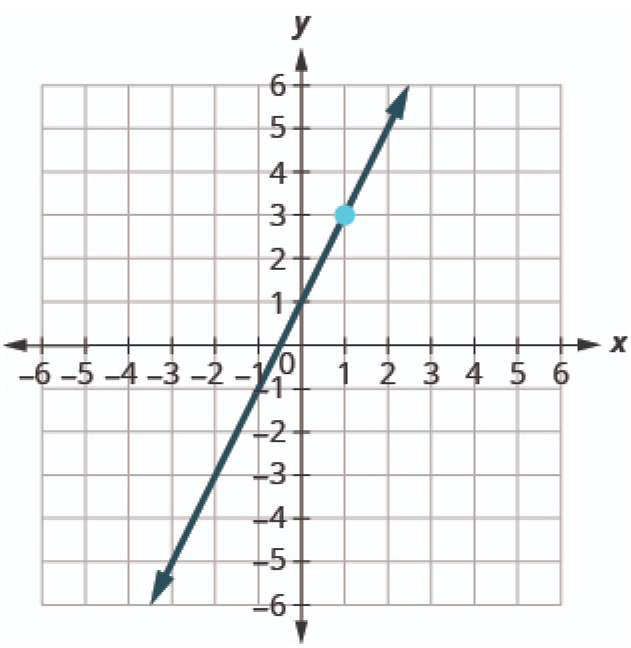
Solution:

$y=-2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

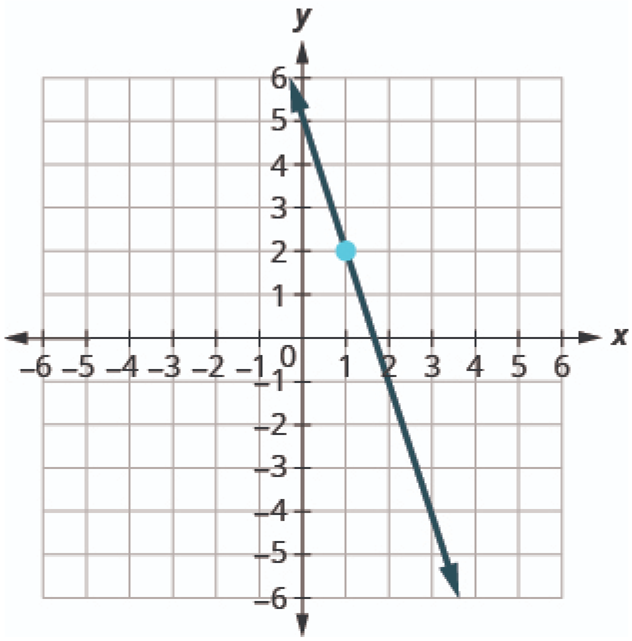
Exercise:

Problem:



Exercise:

Problem:

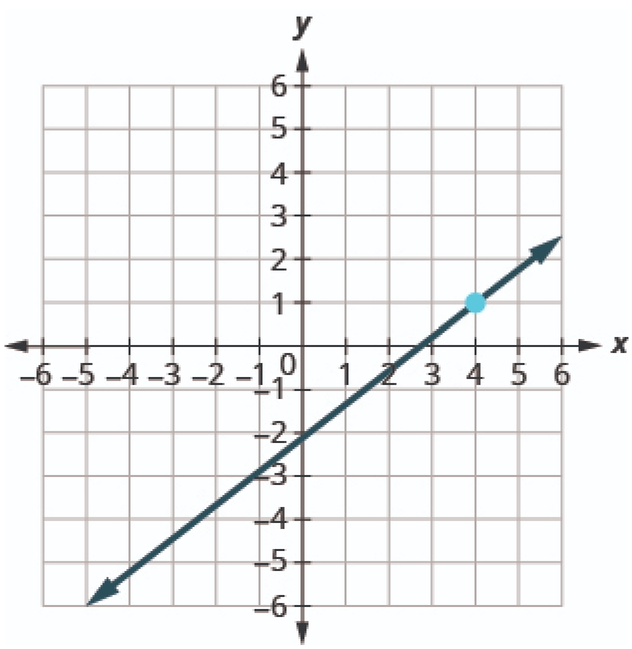


Solution:

$y = -3x + 5$

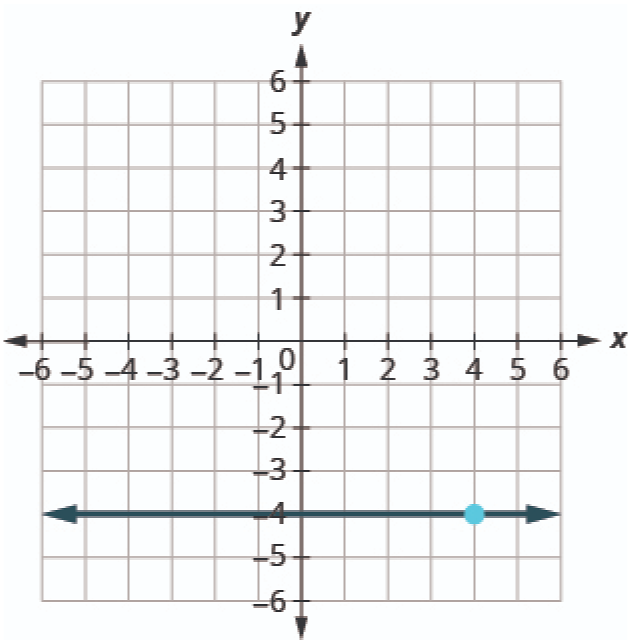
Exercise:

Problem:



Exercise:

Problem:



Solution:

$y = -4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

Exercise:

Problem: $m = -\frac{1}{4}$, point $(-8, 3)$

Exercise:

Problem: $m = \frac{3}{5}$, point $(10, 6)$

Solution:

$y = \frac{3}{5}x$

Exercise:

Problem: Horizontal line containing $(-2, 7)$

Exercise:

Problem: $m = -2$, point $(-1, -3)$

Solution:

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

Exercise:

Problem: $(2, 10)$ and $(-2, -2)$

Exercise:

Problem: $(7, 1)$ and $(5, 0)$

Solution:

$y = \frac{1}{2}x - \frac{5}{2}$

Exercise:

Problem: $(3, 8)$ and $(3, -4)$

Exercise:

Problem: $(5, 2)$ and $(-1, 2)$

Solution:

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise:

Problem: line $y = -3x + 6$, point $(1, -5)$

Exercise:

Problem: line $2x + 5y = -10$, point $(10, 4)$

Solution:

$y = -\frac{2}{5}x + 8$

Exercise:

Problem: line $x = 4$, point $(-2, -1)$

Exercise:

Problem: line $y = -5$, point $(-4, 3)$

Solution:

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–

intercept form.
Exercise:

Problem: line $y=-\frac{4}{5}x+2$, point $(8,9)$

Exercise:

Problem: line $2x-3y=9$, point $(-4,0)$

Solution:

$y=-\frac{3}{2}x-6$

Exercise:

Problem: line $y=3$, point $(-1,-3)$

Exercise:

Problem: line $x=-5$ point $(2,1)$

Solution:

$y=1$

Graph Linear Inequalities in Two Variables

Verify Solutions to an Inequality in Two Variables

In the following exercises, determine whether each ordered pair is a solution to the given inequality.
Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $y < x - 3$:

ⓐ $(0, 1)$ ⓑ $(-2, -4)$ ⓒ $(5, 2)$ ⓓ $(3, -1)$
ⓔ $(-1, -5)$

Exercise:

Problem:

Determine whether each ordered pair is a solution to the inequality $x + y > 4$:

ⓐ $(6, 1)$ ⓑ $(-3, 6)$ ⓒ $(3, 2)$ ⓓ $(-5, 10)$ ⓔ $(0, 0)$

Solution:

ⓐ yes ⓑ no ⓒ yes ⓓ yes; ⓔ no

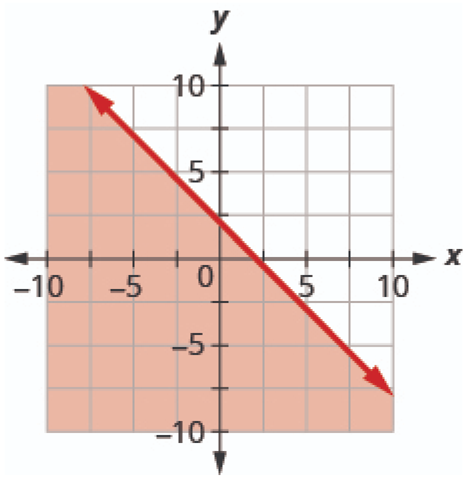
Recognize the Relation Between the Solutions of an Inequality and its Graph

In the following exercises, write the inequality shown by the shaded region.

Exercise:

Problem:

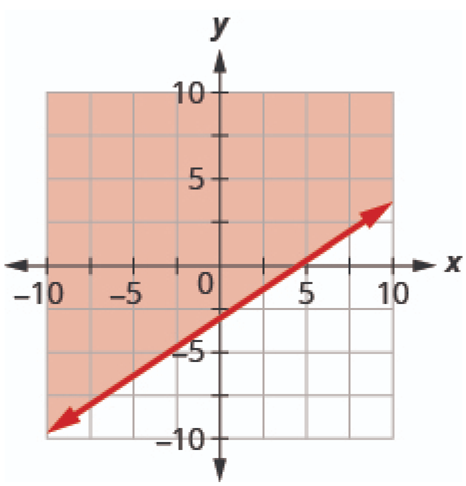
Write the inequality shown by the graph with the boundary line $y = -x + 2$.



Exercise:

Problem:

Write the inequality shown by the graph with the boundary line $y = \frac{2}{3}x - 3$.



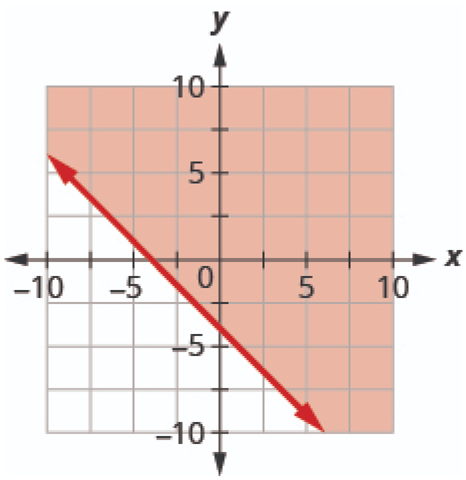
Solution:

$y > \frac{2}{3}x - 3$

Exercise:

Problem:

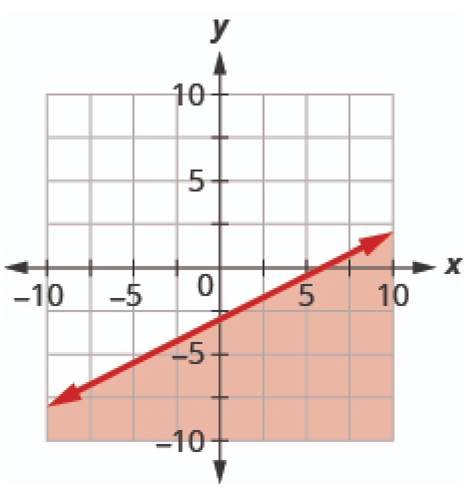
Write the inequality shown by the shaded region in the graph with the boundary line $x + y = -4$.



Exercise:

Problem:

Write the inequality shown by the shaded region in the graph with the boundary line $x - 2y = 6$.



Solution:

$x - 2y \geq 6$

Graph Linear Inequalities in Two Variables

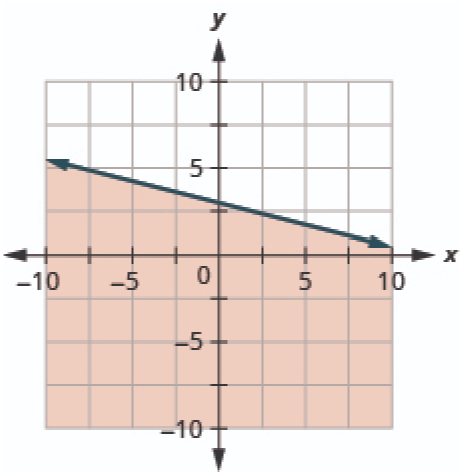
In the following exercises, graph each linear inequality.
Exercise:

Problem: Graph the linear inequality $y > \frac{2}{5}x - 4$.

Exercise:

Problem: Graph the linear inequality $y \leq -\frac{1}{4}x + 3$.

Solution:



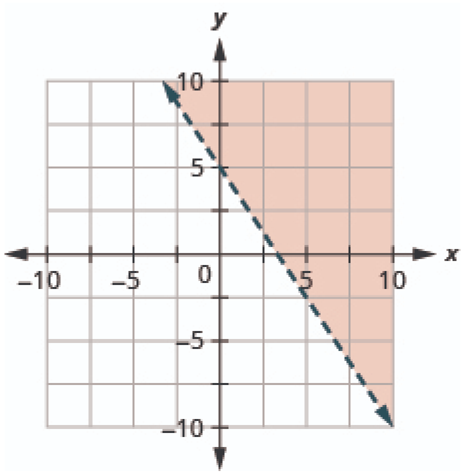
Exercise:

Problem: Graph the linear inequality $x - y \leq 5$.

Exercise:

Problem: Graph the linear inequality $3x + 2y > 10$.

Solution:



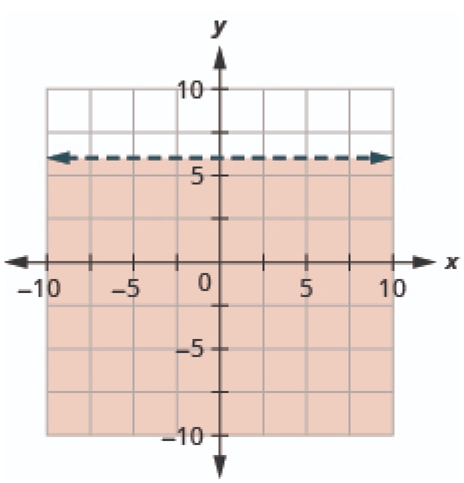
Exercise:

Problem: Graph the linear inequality $y \leq -3x$.

Exercise:

Problem: Graph the linear inequality $y < 6$.

Solution:



Solve Applications using Linear Inequalities in Two Variables

Exercise:

Problem:

Shanthie needs to earn at least $500 a week during her summer break to pay for college. She works two jobs. One as a swimming instructor that pays $10 an hour and the other as an intern in a law office for $25 hour. How many hours does Shanthie need to work at each job to earn at least $500 per week?

ⓐ Let *x* be the number of hours she works teaching swimming and let *y* be the number of hours she works as an intern. Write an inequality that would model this situation.
ⓑ Graph the inequality.
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Shanthie.

Exercise:

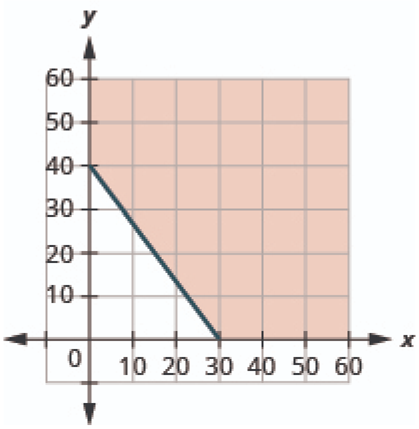
Problem:

Atsushi he needs to exercise enough to burn 600 calories each day. He prefers to either run or bike and burns 20 calories per minute while running and 15 calories a minute while biking.

ⓐ If *x* is the number of minutes that Atsushi runs and *y* is the number minutes he bikes, find the inequality that models the situation.
ⓑ Graph the inequality.
ⓒ List three solutions to the inequality. What options do the solutions provide Atsushi?

Solution:

ⓐ $20x + 15y \geq 600$
ⓑ



ⓒ Answers will vary.

Relations and Functions

Find the Domain and Range of a Relation

In the following exercises, for each relation, ⓐ find the domain of the relation ⓑ find the range of the relation.

Exercise:

Problem: $\{(5, -2), (5, -4), (7, -6), (8, -8), (9, -10)\}$

Exercise:

Problem: $\{(-3, 7), (-2, 3), (-1, 9), (0, -3), (-1, 8)\}$

Solution:

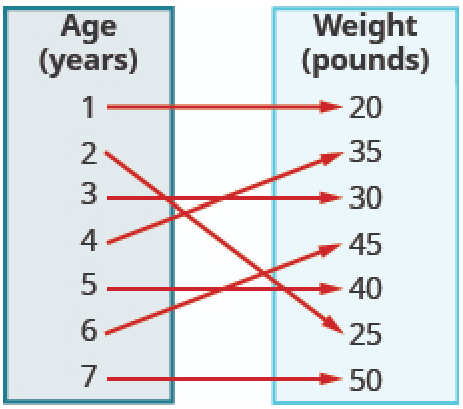
ⓐ D: {−3, −2, −1, 0}
ⓑ R: {7, 3, 9, −3, 8}

In the following exercise, use the mapping of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:

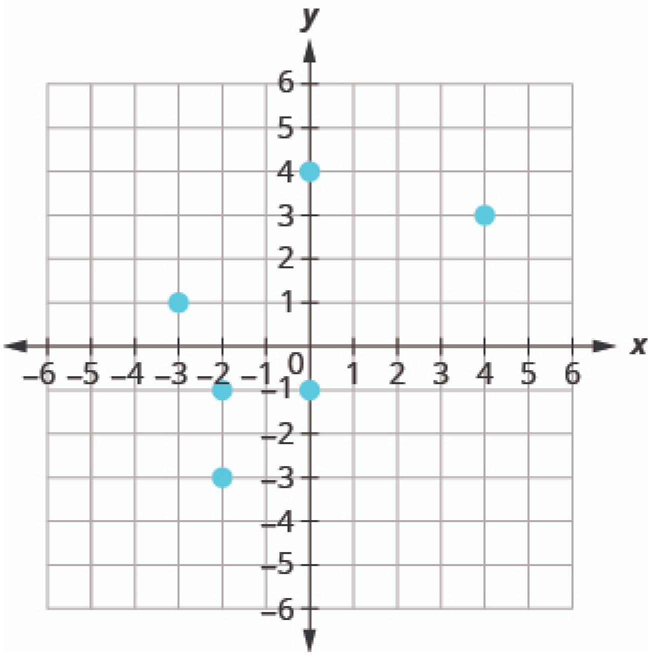
The mapping below shows the average weight of a child according to age.



In the following exercise, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem:



Solution:

ⓐ (4, 3), (−2, −3), (−2, −1), (−3, 1), (0, −1), (0, 4),
ⓑ D: {−3, −2, 0, 4}
ⓒ R: {−3, −1, 1, 3, 4}

Determine if a Relation is a Function

In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.

Exercise:

Problem: $\{(9,-5),(4,-3),(1,-1),(0,0),(1,1),(4,3),(9,5)\}$

Exercise:

Problem: $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$

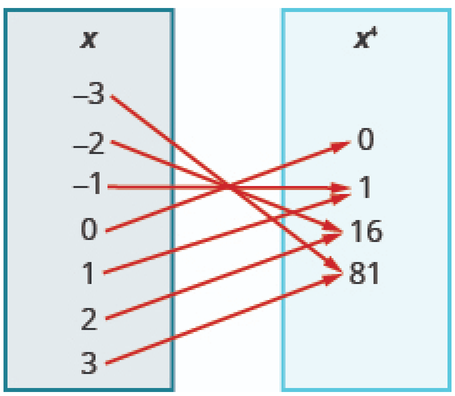
Solution:

ⓐ yes ⓑ {−3, −2, −1, 0, 1, 2, 3}
ⓒ {0, 1, 8, 27}

In the following exercises, use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the function ⓒ find the range of the function.

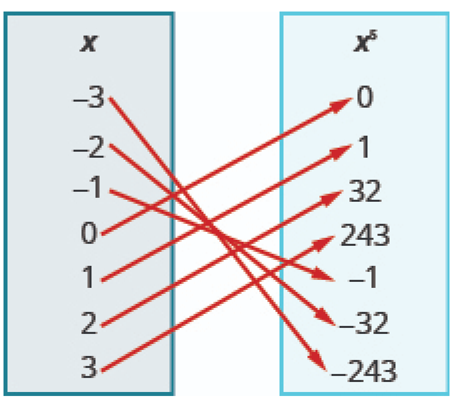
Exercise:

Problem:



Exercise:

Problem:



Solution:

ⓐ {−3, −2, −1, 0, 1, 2, 3}
ⓑ {−3, −2, −1, 0, 1, 2, 3}
ⓒ {−243, −32, −1, 0, 1, 32, 243}

In the following exercises, determine whether each equation is a function.

Exercise:

Problem: $2x + y = -3$

Exercise:

Problem: $y = x^2$

Solution:

yes

Exercise:

Problem: $y = 3x - 5$

Exercise:

Problem: $y = x^3$

Solution:

yes

Exercise:

Problem: $2x + y^2 = 4$

Find the Value of a Function

In the following exercises, evaluate the function:

ⓐ $f(-2)$ ⓑ $f(3)$ ⓒ $f(a)$.

Exercise:

Problem: $f(x) = 3x - 4$

Solution:

ⓐ $f(-2) = -10$ ⓑ $f(3) = 5$ ⓒ $f(a) = 3a - 4$

Exercise:

Problem: $f(x) = -2x + 5$

Exercise:

Problem: $f(x) = x^2 - 5x + 6$

Solution:

ⓐ $f(-2) = 20$ ⓑ $f(3) = 0$ ⓒ $f(a) = a^2 - 5a + 6$

Exercise:

Problem: $f(x) = 3x^2 - 2x + 1$

In the following exercises, evaluate the function.

Exercise:

Problem: $g(x) = 3x^2 - 5x; g(2)$

Solution:

2

Exercise:

Problem: $F(x) = 2x^2 - 3x + 1; F(-1)$

Exercise:

Problem: $h(t) = 4\,|t-1| + 2;\, h(-3)$

Solution:

18

Exercise:

Problem: $f(x) = \frac{x+2}{x-1};\, f(3)$

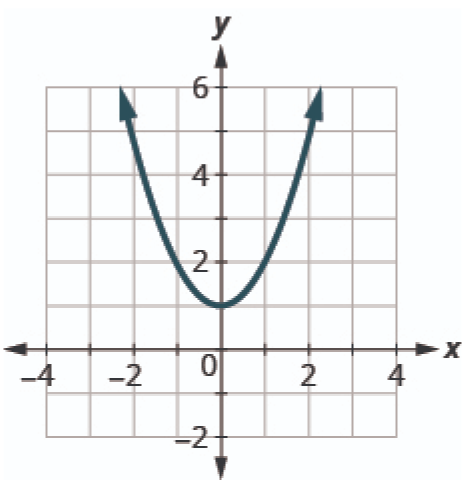
Graphs of Functions

Use the Vertical line Test

In the following exercises, determine whether each graph is the graph of a function.

Exercise:

Problem:

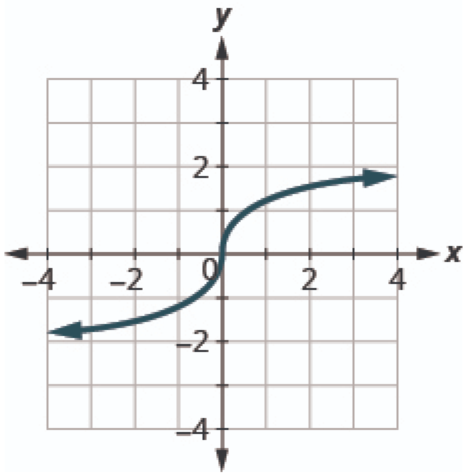


Solution:

yes

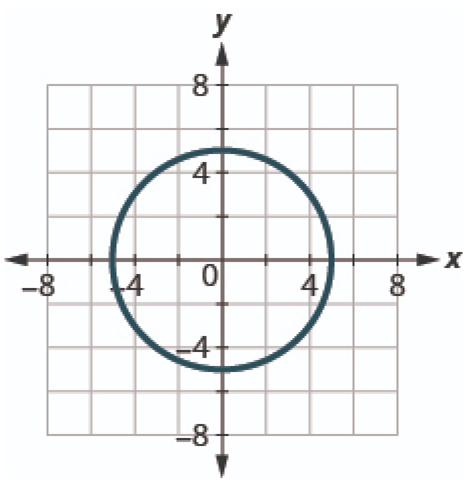
Exercise:

Problem:



Exercise:

Problem:

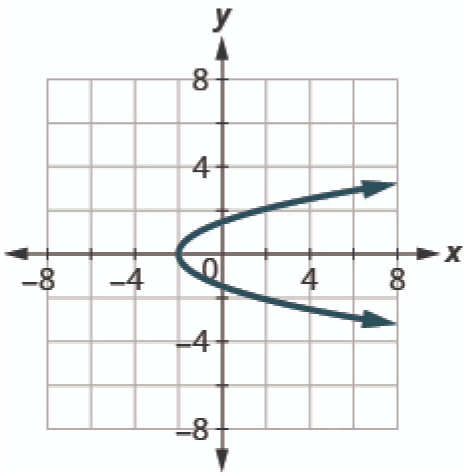


Solution:

no

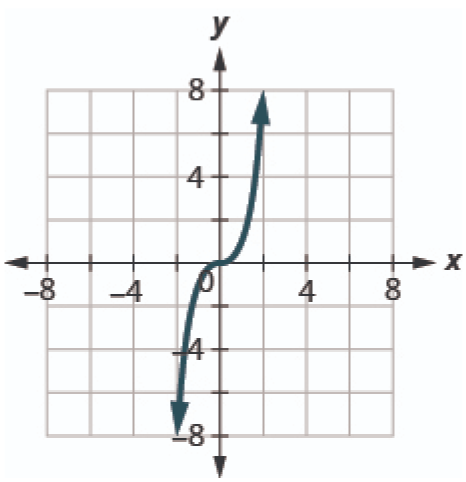
Exercise:

Problem:



Exercise:

Problem:

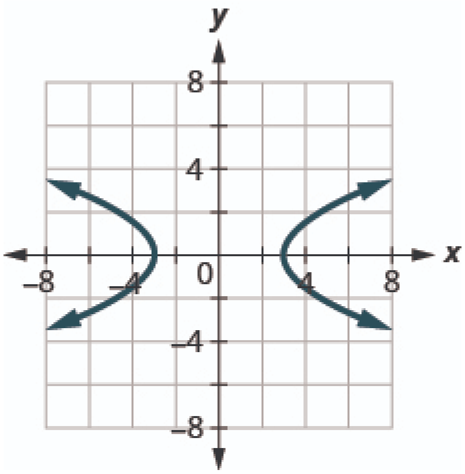


Solution:

yes

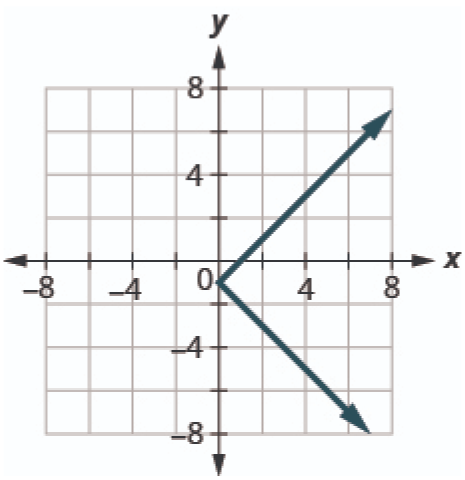
Exercise:

Problem:



Exercise:

Problem:



Solution:

no

Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

Exercise:

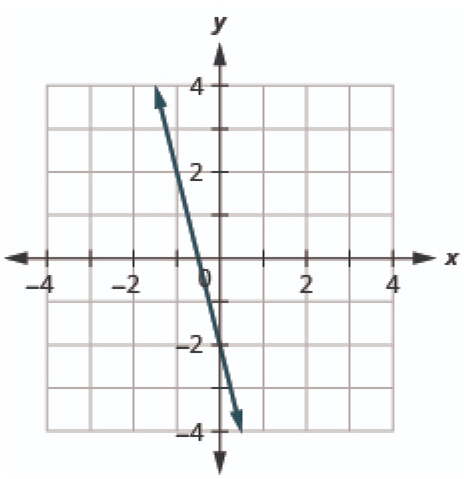
Problem: $f(x) = 5x + 1$

Exercise:

Problem: $f(x) = -4x - 2$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

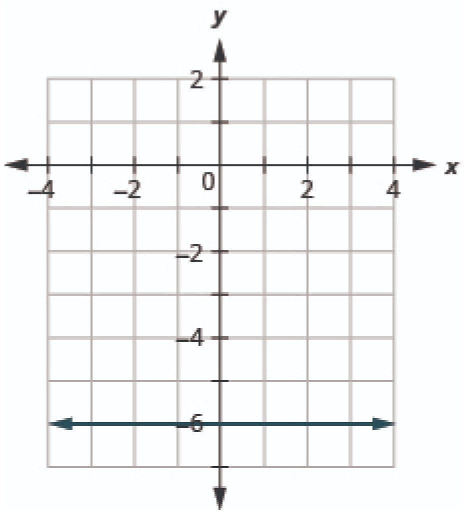
Problem: $f(x) = \frac{2}{3}x - 1$

Exercise:

Problem: $f(x) = -6$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

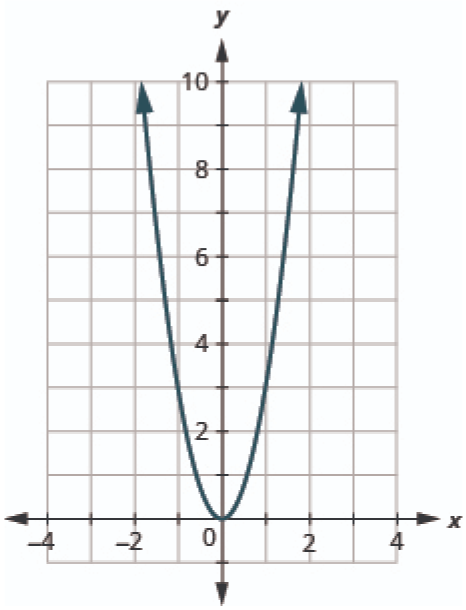
Problem: $f\left(x\right)=2x$

Exercise:

Problem: $f\left(x\right)=3{x}^{2}$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,0]

Exercise:

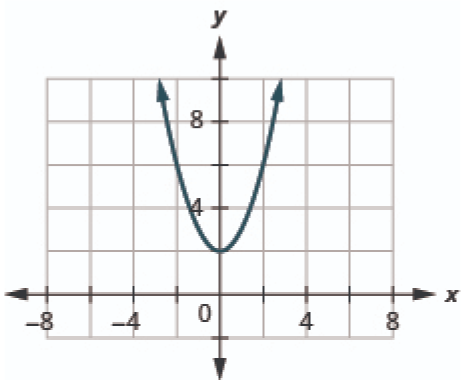
Problem: $f(x) = -\frac{1}{2}x^2$

Exercise:

Problem: $f(x) = x^2 + 2$

Solution:

ⓐ



ⓑ D: (-∞,∞), R: (-∞,∞)

Exercise:

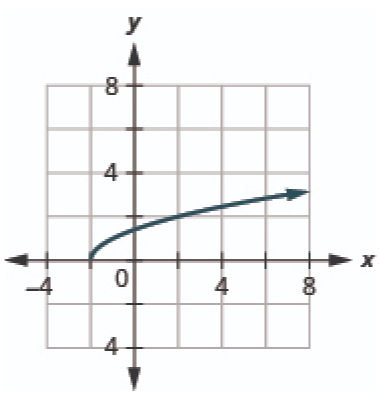
Problem: $f\left(x\right)={x}^{3}-2$

Exercise:

Problem: $f\left(x\right)=\sqrt{x+2}$

Solution:

ⓐ



ⓑ D: $\left[-2,\infty \right)$, R: [0,∞)

Exercise:

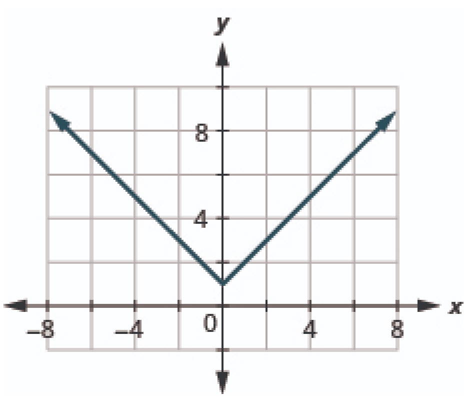
Problem: $f\left(x\right)=-\left|x\right|$

Exercise:

Problem: $f\left(x\right)=\left|x\right|+1$

Solution:

ⓐ



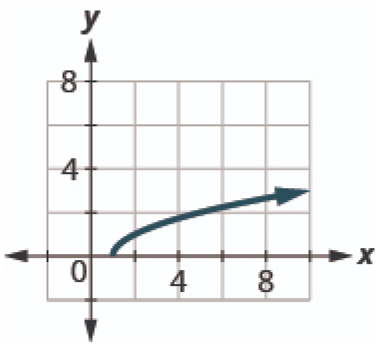
ⓑ D: (-∞,∞), R: [1,∞)

Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation

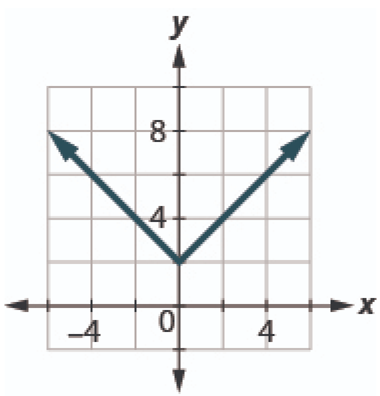
Exercise:

Problem:



Exercise:

Problem:

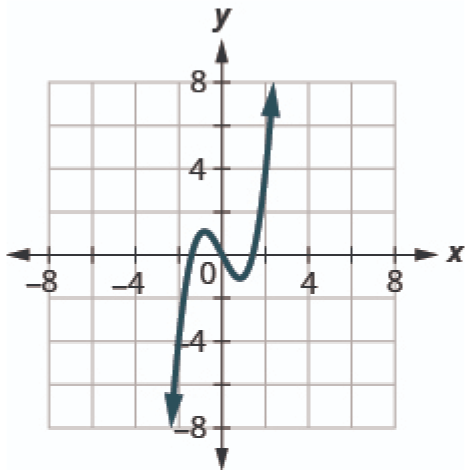


Solution:

D: (-∞,∞), R: [2,∞)

Exercise:

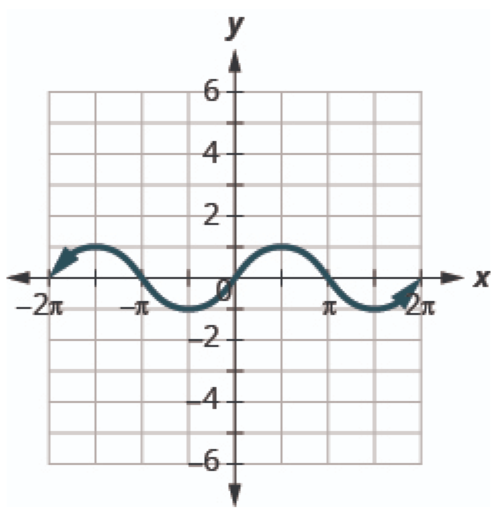
Problem:



In the following exercises, use the graph of the function to find the indicated values.

Exercise:

Problem:



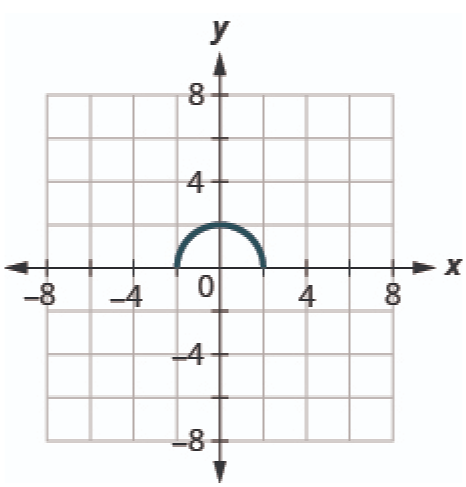
ⓐ Find $f(0)$.
ⓑ Find $f\left(\frac{1}{2}\pi\right)$.
ⓒ Find $f\left(-\frac{3}{2}\pi\right)$.
ⓓ Find the values for x when $f(x) = 0$.
ⓔ Find the x-intercepts.
ⓕ Find the y-intercepts.
ⓖ Find the domain. Write it in interval notation.
ⓗ Find the range. Write it in interval notation.

Solution:

ⓐ $f(x) = 0$ ⓑ $f(\pi/2) = 1$
ⓒ $f(-3\pi/2) = 1$ ⓓ $f(x) = 0$ for $x = -2\pi, -\pi, 0, \pi, 2\pi$
ⓔ $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$ (f) $(0, 0)$
ⓖ $[-2\pi, 2\pi]$ ⓗ $[-1, 1]$

Exercise:

Problem:



ⓐ Find $f(0)$.
ⓑ Find the values for x when $f(x) = 0$.
ⓒ Find the x-intercepts.
ⓓ Find the y-intercepts.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

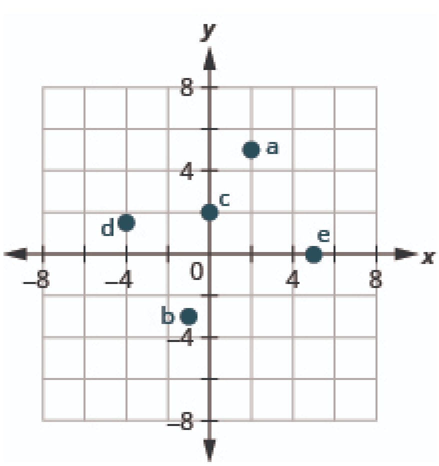
Practice Test

Exercise:

Problem: Plot each point in a rectangular coordinate system.

ⓐ $(2, 5)$
ⓑ $(-1, -3)$
ⓒ $(0, 2)$
ⓓ $\left(-4, \frac{3}{2}\right)$
ⓔ $(5, 0)$

Solution:



Exercise:

Problem:

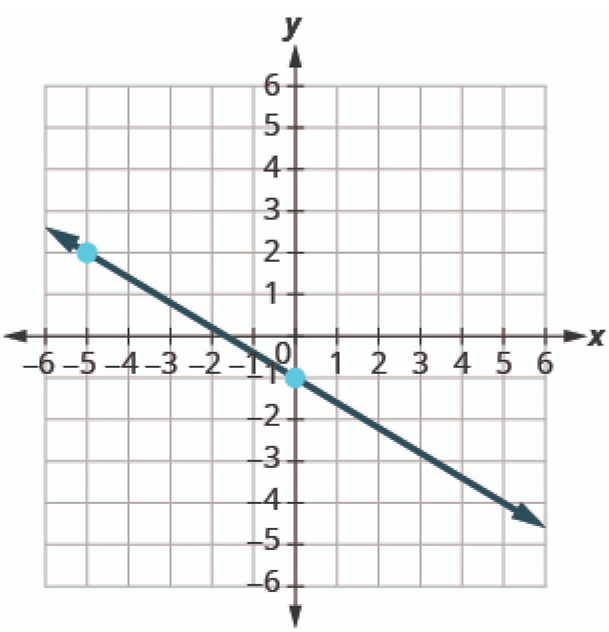
Which of the given ordered pairs are solutions to the equation $3x - y = 6$?

ⓐ $(3, 3)$ ⓑ $(2, 0)$ ⓒ $(4, -6)$

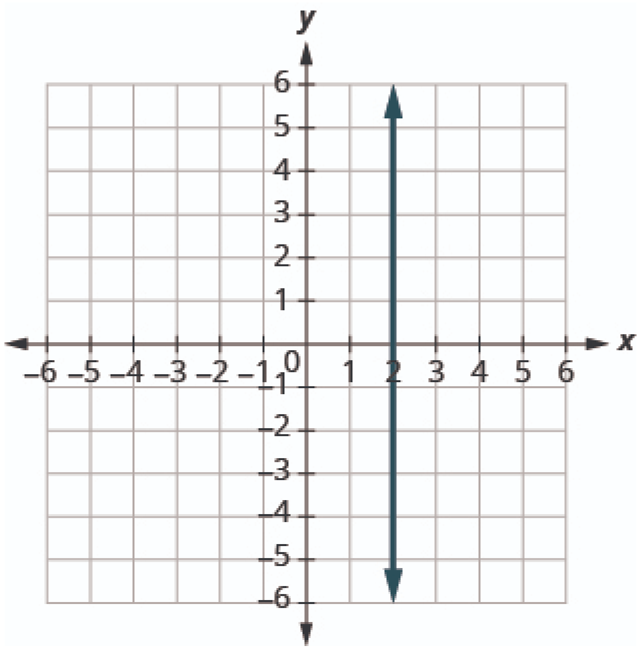
Find the slope of each line shown.

Exercise:

Problem: ⓐ



ⓑ



Solution:

ⓐ $-\frac{3}{5}$ ⓑ undefined

Exercise:

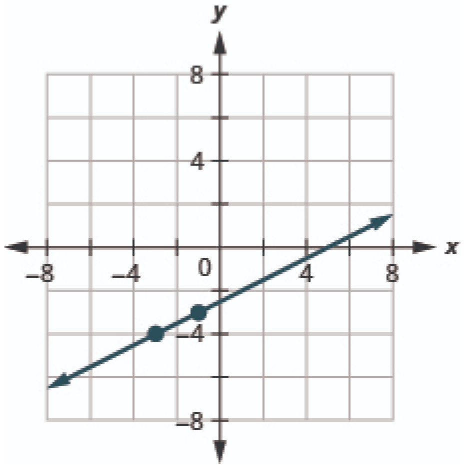
Problem:

Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

Exercise:

Problem: Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

Solution:



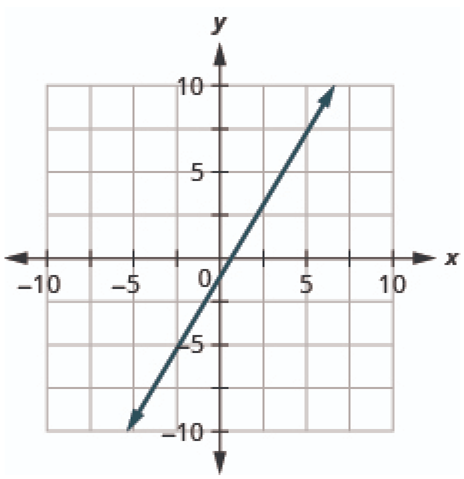
Exercise:

Problem: Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.
Exercise:

Problem: $y = \frac{5}{3}x - 1$

Solution:



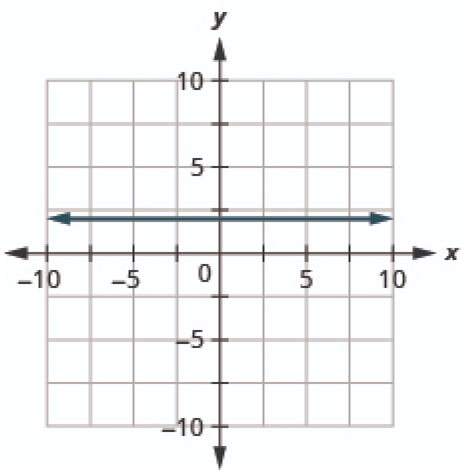
Exercise:

Problem: $y = -x$

Exercise:

Problem: $y = 2$

Solution:



Find the equation of each line. Write the equation in slope-intercept form.

Exercise:

Problem: slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

Exercise:

Problem: $m = 2$, point $(-3, -1)$

Solution:

$y = 2x + 5$

Exercise:

Problem: containing $(10, 1)$ and $(6, -1)$

Exercise:

Problem:

perpendicular to the line $y=\frac{5}{4}x+2$, containing the point $(-10,3)$

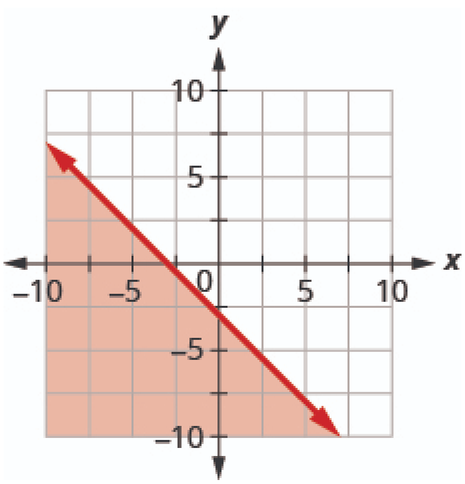
Solution:

$y=-\frac{4}{5}x-5$

Exercise:

Problem:

Write the inequality shown by the graph with the boundary line $y=-x-3$.

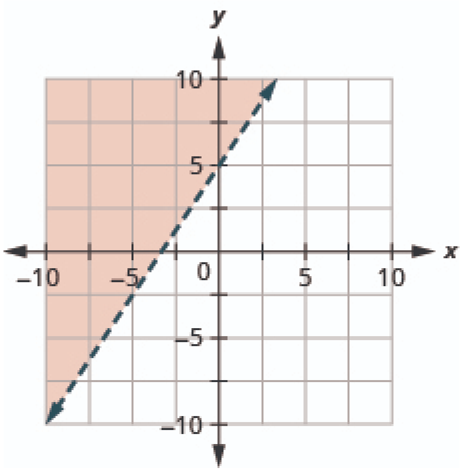


Graph each linear inequality.

Exercise:

Problem: $y>\frac{3}{2}x+5$

Solution:



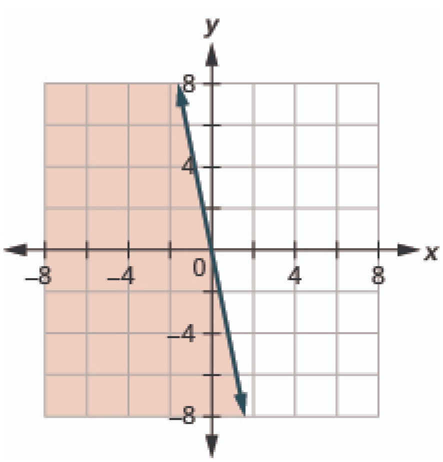
Exercise:

Problem: $x - y \geq -4$

Exercise:

Problem: $y \leq -5x$

Solution:



Exercise:

Problem:

Hiro works two part time jobs in order to earn enough money to meet her obligations of at least $450 a week. Her job at the mall pays $10 an hour and her administrative assistant job on campus pays $15 an hour. How many hours does Hiro need to work at each job to earn at least $450?

ⓐ Let *x* be the number of hours she works at the mall and let *y* be the number of hours she works as administrative assistant. Write an inequality that would model this situation.
ⓑ Graph the inequality .
ⓒ Find three ordered pairs (x, y) that would be solutions to the inequality. Then explain what that means for Hiro.

Exercise:

Problem:

Use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

$\{(-3, 27), (-2, 8), (-1, 1), (0, 0),$
$(1, 1), (2, 8), (3, 27)\}$

Solution:

ⓐ yes ⓑ $\{-3, -2, -1, 0, 1, 2, 3\}$ ⓒ {0, 1, 8, 27}

Exercise:

Problem: Evaluate the function: ⓐ $f(-1)$ ⓑ $f(2)$ ⓒ $f(c)$.

$f(x) = 4x^2 - 2x - 3$

Exercise:

Problem: For $h(y) = 3\left|y - 1\right| - 3$, evaluate $h(-4)$.

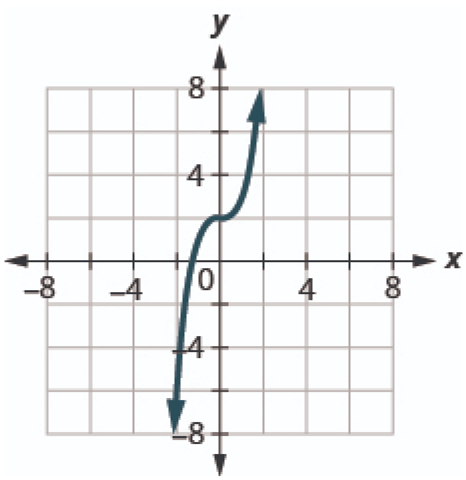
Solution:

12

Exercise:

Problem:

Determine whether the graph is the graph of a function. Explain your answer.



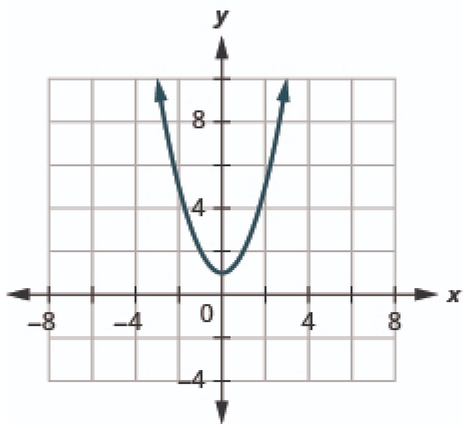
In the following exercises, ⓐ graph each function ⓑ state its domain and range.
Write the domain and range in interval notation.

Exercise:

Problem: $f(x) = x^2 + 1$

Solution:

ⓐ



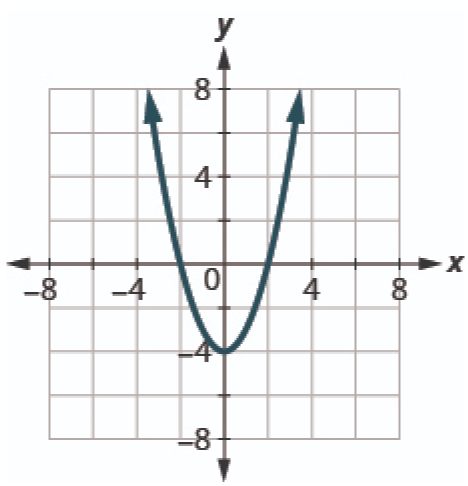
ⓑ D: (-∞,∞), R: [1,∞)

Exercise:

Problem: $f\left(x\right)=\sqrt{x+1}$

Exercise:

Problem:



ⓑ Find the y-intercepts.
ⓒ Find $f(-1)$.
ⓓ Find $f(1)$.
ⓔ Find the domain. Write it in interval notation.
ⓕ Find the range. Write it in interval notation.

Solution:

ⓐ $x = -2, 2$ ⓑ $y = -4$
ⓒ $f(-1) = -3$ ⓓ $f(1) = -3$
ⓔ D: (-∞,∞) ⓕ R: $[-4, \infty)$

Introduction
class="introduction"

American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: “keijj44” / Pixabay)

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.

Multiply and Divide Rational Expressions
By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review [link].
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.
   If you missed this problem, review [link].

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Note:
Rational Expression
A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:
Equation:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Note:
Determine the values for which a rational expression is undefined.

Set the denominator equal to zero.
Solve the equation.

Example:
Exercise:

Problem: Determine the value for which each rational expression is undefined:

ⓐ $\frac{8a^2b}{3c}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$.

Solution:

The expression will be undefined when the denominator is zero.

ⓐ

Set the denominator equal to zero and solve for the variable.

$$\frac{8a^2b}{3c}$$

$$3c = 0$$

$$c = 0$$

$\frac{8a^2b}{3c}$ is undefined for $c = 0$.

ⓑ

Set the denominator equal to zero and solve for the variable.

$$\frac{4b-3}{2b+5}$$

$$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$$

$\frac{4b-3}{2b+5}$ is undefined for $b = -\frac{5}{2}$.

ⓒ

Set the denominator equal to zero and solve for the variable.

$$\frac{x+4}{x^2+5x+6}$$
$$x^2+5x+6=0$$
$$(x+2)(x+3)=0$$
$$x+2=0 \text{ or } x+3=0$$
$$x=-2 \text{ or } x=-3$$

$\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$.

Note:
Exercise:

Problem: Determine the value for which each rational expression is undefined.

ⓐ $\frac{3y^2}{8x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

Solution:

ⓐ $x=0$ ⓑ $n=-\frac{1}{3}$
ⓒ $a=-1, a=-3$

Note:
Exercise:

Problem: Determine the value for which each rational expression is undefined.

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Solution:

ⓐ $q=0$ ⓑ $y=-\frac{2}{3}$
ⓒ $m=2, m=-3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Note:

Simplified Rational Expression
A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,
Equation:

$$\frac{x+2}{x+3} \text{ is simplified because there are no common factors of } x+2 \text{ and } x+3.$$
$$\frac{2x}{3x} \text{ is not simplified because } x \text{ is a common factor of } 2x \text{ and } 3x.$$

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Note:
Equivalent Fractions Property
If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0,$
Equation:

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\dfrac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\dfrac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$	$\dfrac{x+5}{x}$
$\dfrac{2}{5}$	$\dfrac{3x}{5}$	NO COMMON FACTORS
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the *x*'s from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

Example:
How to Simplify a Rational Expression
Exercise:

Problem: Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution:

Step 1. Factor the numerator and denominator completely.	Factor $x^2 + 5x + 6$ and $x^2 + 8x + 12$.	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x + 2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

Note:
Exercise:

Problem: Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Solution:

$\frac{x+1}{x-1}, x \neq 2, x \neq 1$

Note:
Exercise:

Problem: Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Solution:

$\frac{x-5}{x-1}, x \neq -2, x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

Note:
Simplify a rational expression.

Factor the numerator and denominator completely.
Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Example:
Exercise:

Problem: Simplify: $\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution:

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2}$$

Factor the numerator and denominator, first factoring out the GCF.

$$\frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

Remove the common factors of $a-2b$ and 3.

$$\frac{\cancel{3}(a-2b)\cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

Note:
Exercise:

Problem: Simplify: $\frac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Solution:

$\frac{2(x-3y)}{3(x+3y)}$

Note:
Exercise:

Problem: Simplify: $\frac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Solution:

$\frac{5(x-y)}{2(x+5y)}$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.
Equation:

$$\begin{array}{ll} \text{Let's look at the expression } b-a. & b-a \\ \text{Rewrite.} & -a+b \\ \text{Factor out } -1. & -1\left(a-b\right) \end{array}$$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Note:
Opposites in a Rational Expression
The opposite of $a-b$ is $b-a$.
Equation:

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a}=1$.

Example:
Exercise:

Problem: Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution:

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2-4x-5}{25-x^2}$.

Solution:

$-\frac{x+1}{x+5}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2+x-2}{1-x^2}$.

Solution:

$-\frac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Note:
Multiplication of Rational Expressions
If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then
Equation:

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$,$x \neq 3$, and $x \neq 4$.

Example:
How to Multiply Rational Expressions
Exercise:

Problem: Simplify: $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Solution:

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$

Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot\cancel{x}\cdot x\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

Note:
Exercise:

Problem: Simplify: $\frac{5x}{x^2+5x+6}\cdot\frac{x^2-4}{10x}$.

Solution:

$\frac{x-2}{2(x+3)}$

Note:
Exercise:

Problem: Simplify: $\frac{9x^2}{x^2+11x+30}\cdot\frac{x^2-36}{3x^2}$.

Solution:

$\frac{3(x-6)}{x+5}$

Note:
Multiply rational expressions.

Factor each numerator and denominator completely.
Multiply the numerators and denominators.
Simplify by dividing out common factors.

Example:
Exercise:

Problem: Multiply: $\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$.

Solution:

	$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5}$
Factor the numerators and denominators and then multiply.	$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$
Simplify.	$\frac{(a-3)(a+5)}{(a-5)(a-5)}$
Rewrite $(a-5)(a-5)$ using an exponent.	$\frac{(a-3)(a+5)}{(a-5)^2}$

Note:
Exercise:

Problem: Simplify: $\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$.

Solution:

$\frac{x-4}{x-5}$

Note:
Exercise:

Problem: Simplify: $\frac{4b^2+7b-2}{1-b^2} \cdot \frac{b^2-2b+1}{4b^2+15b-4}$.

Solution:

$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Note:
Division of Rational Expressions
If p, q, r, and s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then
Equation:

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

Example:
How to Divide Rational Expressions
Exercise:

Problem: Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution:

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Note:
Exercise:

Problem: Simplify: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Solution:

$\frac{2\left(x^2+2x+4\right)}{(x+2)(x^2-2x+4)}$

Note:
Exercise:

Problem: Simplify: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3+1}$.

Solution:

$\frac{2z}{z-1}$

Note:
Divide rational expressions.

Rewrite the division as the product of the first rational expression and the reciprocal of the second.
Factor the numerators and denominators completely.
Multiply the numerators and denominators together.
Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Example:
Exercise:

Problem: Divide: $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign.	$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of first times reciprocal of second.	$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply.	$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify by dividing out common factors.	$\frac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$
Simplify.	$\frac{3x-2}{4}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

Solution:

$\frac{x+2}{4}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

Solution:

$\frac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

Example:
Exercise:

Problem: Perform the indicated operations: $\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Solution:

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)} \cdot \frac{(x+3)(x-1)}{(x+2)(x-5)} \cdot \frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4} \cdot 2\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Note:
Exercise:

Problem: Perform the indicated operations: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

Solution:

$\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Note:
Exercise:

Problem: Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Solution:

$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$

Multiply and Divide Rational Functions

We started this section stating that a rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

Note:
Rational Function
A rational function is a function of the form
Equation:

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

Note:
Determine the domain of a rational function.

Set the denominator equal to zero.
Solve the equation.
The domain is all real numbers excluding the values found in Step 2.

Example:
Exercise:

Problem: Find the domain of $R(x) = \frac{2x^2-14x}{4x^2-16x-48}$.

Solution:

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2 - 16x - 48 = 0$
Factor, first factor out the GCF.	$4\left(x^2 - 4x - 12\right) = 0$
	$4(x-6)(x+2) = 0$
Use the Zero Product Property.	$4 \neq 0 \quad x - 6 = 0 \quad x + 2 = 0$
Solve.	$x = 6 \quad x = -2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$.

Note:
Exercise:

Problem: Find the domain of $R(x) = \frac{2x^2-10x}{4x^2-16x-20}$.

Solution:

The domain of $R(x)$ is all real numbers where $x \neq 5$ and $x \neq -1$.

Note:
Exercise:

Problem: Find the domain of $R(x) = \frac{4x^2-16x}{8x^2-16x-64}$.

Solution:

The domain of $R(x)$ is all real numbers where $x \neq 4$ and $x \neq -2$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

Example:

Exercise:

Problem: Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{2x-6}{x^2-8x+15}$ and $g(x) = \frac{x^2-25}{2x+10}$.

Solution:

$$R(x) = f(x) \cdot g(x)$$

$$R(x) = \frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$$

Factor each numerator and denominator.

$$R(x) = \frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$$

Multiply the numerators and denominators.

$$R(x) = \frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$$

Remove common factors.

$$R(x) = \frac{\cancel{2}\ \cancel{(x-3)}\ \cancel{(x-5)}\ \cancel{(x+5)}}{\cancel{2}\ \cancel{(x-3)}\ \cancel{(x-5)}\ \cancel{(x+5)}}$$

Simplify.

$$R(x) = 1$$

Note:
Exercise:

Problem: Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{3x-21}{x^2-9x+14}$ and $g(x) = \frac{2x^2-8}{3x+6}$.

Solution:

$R(x) = 2$

Note:
Exercise:

Problem: Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{x^2-x}{3x^2+27x-30}$ and $g(x) = \frac{x^2-100}{x^2-10x}$.

Solution:

$R(x) = \frac{1}{3}$

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

Example:
Exercise:

Problem: Find $R(x) = \frac{f(x)}{g(x)}$ where $f(x) = \frac{3x^2}{x^2-4x}$ and $g(x) = \frac{9x^2-45x}{x^2-7x+10}$.

Solution:

$$R(x) = \frac{f(x)}{g(x)}$$

Substitute in the functions $f(x),\ g(x)$.

$$R(x) = \frac{\frac{3x^2}{x^2-4x}}{\frac{9x^2-45x}{x^2-7x+10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x) = \frac{3x^2}{x^2-4x} \cdot \frac{x^2-7x+10}{9x^2-45x}$$

Factor the numerators and denominators and then multiply.

$$R(x) = \frac{3 \cdot x \cdot x \cdot (x-5)(x-2)}{x(x-4) \cdot 3 \cdot 3 \cdot x \cdot (x-5)}$$

Simplify by dividing out common factors.

$$R(x) = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x}\ \cancel{(x-5)}\ (x-2)}{\cancel{x}\ (x-4) \cdot \cancel{3} \cdot 3 \cdot \cancel{x}\ \cancel{(x-5)}}$$

$$R(x) = \frac{x-2}{3(x-4)}$$

Note:
Exercise:

Problem: Find $R(x) = \frac{f(x)}{g(x)}$ where $f(x) = \frac{2x^2}{x^2-8x}$ and $g(x) = \frac{8x^2+24x}{x^2+x-6}$.

Solution:

$R(x) = \frac{x-2}{4(x-8)}$

Note:
Exercise:

Problem: Find $R(x) = \frac{f(x)}{g(x)}$ where $f(x) = \frac{15x^2}{3x^2+33x}$ and $g(x) = \frac{5x-5}{x^2+9x-22}$.

Solution:

$R(x) = \frac{x(x-2)}{x-1}$

Key Concepts

- **Determine the values for which a rational expression is undefined.**

  Set the denominator equal to zero.
  Solve the equation.

- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **How to simplify a rational expression.**

  Factor the numerator and denominator completely.
  Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.
  $\frac{a-b}{b-a} = -1 \qquad a \neq b$
  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0, s \neq 0$, then
  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**

  Factor each numerator and denominator completely.
  Multiply the numerators and denominators.
  Simplify by dividing out common factors.

- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then
  $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$
- **How to divide rational expressions.**

  Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  Factor the numerators and denominators completely.
  Multiply the numerators and denominators together.
  Simplify by dividing out common factors.

- **How to determine the domain of a rational function.**

  Set the denominator equal to zero.
  Solve the equation.
  The domain is all real numbers excluding the values found in Step 2.

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.
Exercise:

Problem: ⓐ $\frac{2x^2}{z}$
ⓑ $\frac{4p-1}{6p-5}$
ⓒ $\frac{n-3}{n^2+2n-8}$

Solution:

ⓐ $z=0$ ⓑ $p=\frac{5}{6}$
ⓒ $n=-4, n=2$

Exercise:

Problem: ⓐ $\frac{10m}{11n}$
ⓑ $\frac{6y+13}{4y-9}$
ⓒ $\frac{b-8}{b^2-36}$

Exercise:

Problem: ⓐ $\frac{4x^2y}{3y}$
ⓑ $\frac{3x-2}{2x+1}$
ⓒ $\frac{u-1}{u^2-3u-28}$

Solution:

ⓐ $y=0$ ⓑ $x=-\frac{1}{2}$
ⓒ $u=-4, u=7$

Exercise:

Problem: ⓐ $\frac{5pq^2}{9q}$
ⓑ $\frac{7a-4}{3a+5}$
ⓒ $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

Exercise:

Problem: $-\frac{44}{55}$

Solution:

$-\frac{4}{5}$

Exercise:

Problem: $\frac{56}{63}$

Exercise:

Problem: $\frac{8m^3n}{12mn^2}$

Solution:

$\frac{2m^2}{3n}$

Exercise:

Problem: $\frac{36v^3w^2}{27vw^3}$

Exercise:

Problem: $\frac{8n-96}{3n-36}$

Solution:

$\frac{8}{3}$

Exercise:

Problem: $\frac{12p-240}{5p-100}$

Exercise:

Problem: $\frac{x^2+4x-5}{x^2-2x+1}$

Solution:

$\frac{x+5}{x-1}$

Exercise:

Problem: $\frac{y^2+3y-4}{y^2-6y+5}$

Exercise:

Problem: $\frac{a^2-4}{a^2+6a-16}$

Solution:

$\frac{a+2}{a+8}$

Exercise:

Problem: $\frac{y^2-2y-3}{y^2-9}$

Exercise:

Problem: $\frac{p^3+3p^2+4p+12}{p^2+p-6}$

Solution:

$\frac{p^2+4}{p-2}$

Exercise:

Problem: $\frac{x^3-2x^2-25x+50}{x^2-25}$

Exercise:

Problem: $\frac{8b^2-32b}{2b^2-6b-80}$

Solution:

$\frac{4b(b-4)}{(b+5)(b-8)}$

Exercise:

Problem: $\frac{-5c^2-10c}{-10c^2+30c+100}$

Exercise:

Problem: $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

Solution:

$\frac{3(m+5n)}{4(m-5n)}$

Exercise:

Problem: $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

Exercise:

Problem: $\frac{a-5}{5-a}$

Solution:

-1

Exercise:

Problem: $\frac{5-d}{d-5}$

Exercise:

Problem: $\frac{20-5y}{y^2-16}$

Solution:

$-\frac{5}{y+4}$

Exercise:

Problem: $\frac{4v-32}{64-v^2}$

Exercise:

Problem: $\frac{w^3+216}{w^2-36}$

Solution:

$\frac{w^2-6w+36}{w-6}$

Exercise:

Problem: $\frac{v^3+125}{v^2-25}$

Exercise:

Problem: $\frac{z^2-9z+20}{16-z^2}$

Solution:

$-\frac{z-5}{4+z}$

Exercise:

Problem: $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

Exercise:

Problem: $\frac{12}{16} \cdot \frac{4}{10}$

Solution:

$\frac{3}{10}$

Exercise:

Problem: $\frac{32}{5} \cdot \frac{16}{24}$

Exercise:

Problem: $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Solution:

$\frac{x^3}{8y}$

Exercise:

Problem: $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

Exercise:

Problem: $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Solution:

$\frac{p(p-4)}{2(p-9)}$

Exercise:

Problem: $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

Exercise:

Problem: $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Solution:

$\frac{y-5}{3(y+5)}$

Exercise:

Problem: $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

Exercise:

Problem: $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Solution:

$-\frac{4(b+9)}{3(b+7)}$

Exercise:

Problem: $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

Exercise:

Problem: $\frac{5c^2+9c+2}{c^2-25} \cdot \frac{c^2+10c+25}{3c^2-14c-5}$

Solution:

$\frac{(c+2)(c+2)}{(c-2)(c-3)}$

Exercise:

Problem: $\frac{2d^2+d-3}{d^2-16} \cdot \frac{d^2-8d+16}{2d^2-9d-18}$

Exercise:

Problem: $\frac{6m^2-2m-10}{9-m^2} \cdot \frac{m^2-6m+9}{6m^2+29m-20}$

Solution:

$-\frac{(m-2)(m-3)}{(3+m)(m+4)}$

Exercise:

Problem: $\frac{2n^2-3n-14}{25-n^2} \cdot \frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

Exercise:

Problem: $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Solution:

$-\frac{1}{v+5}$

Exercise:

Problem: $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

Exercise:

Problem: $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Solution:

$\frac{3s}{s+4}$

Exercise:

Problem: $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

Exercise:

Problem: $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Solution:

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Exercise:

Problem: $\frac{v^3-8w^3}{2v^2+4vw+8w^2} \div \frac{v^2-4w^2}{4}$

Exercise:

Problem: $\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Solution:

$\frac{x-2}{8x}$

Exercise:

Problem: $\frac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

Exercise:

Problem: $\frac{\frac{2a^2-a-21}{5a+20}}{\frac{a^2+7a+12}{a^2+8a+16}}$

Solution:

$\frac{2a-7}{5}$

Exercise:

Problem: $\frac{\frac{3b^2+2b-8}{12b+18}}{\frac{3b^2+2b-8}{2b^2-7b-15}}$

Exercise:

Problem: $\frac{\frac{12c^2-12}{2c^2-3c+1}}{\frac{4c+4}{6c^2-13c+5}}$

Solution:

$3\left(3c-5\right)$

Exercise:

Problem: $\frac{\frac{4d^2+7d-2}{35d+10}}{\frac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

Exercise:

Problem: $\frac{10m^2+80m}{3m-9}\cdot\frac{m^2+4m-21}{m^2-9m+20}\div\frac{5m^2+10m}{2m-10}$

Solution:

$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$

Exercise:

Problem: $\frac{4n^2+32n}{3n+2}\cdot\frac{3n^2-n-2}{n^2+n-30}\div\frac{108n^2-24n}{n+6}$

Exercise:

Problem: $\frac{12p^2+3p}{p+3}\div\frac{p^2+2p-63}{p^2-p-12}\cdot\frac{p-7}{9p^3-9p^2}$

Solution:

$\frac{(4p+1)(p-7)}{3p(p+9)(p-1)}$

Exercise:

Problem: $\frac{6q+3}{9q^2-9q}\div\frac{q^2+14q+33}{q^2+4q-5}\cdot\frac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

Exercise:

Problem: $R\left(x\right)=\frac{x^3-2x^2-25x+50}{x^2-25}$

Solution:

$x \neq 5$ and $x \neq -5$

Exercise:

Problem: $R(x) = \frac{x^3+3x^2-4x-12}{x^2-4}$

Exercise:

Problem: $R(x) = \frac{3x^2+15x}{6x^2+6x-36}$

Solution:

$x \neq 2$ and $x \neq -3$

Exercise:

Problem: $R(x) = \frac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x) = f(x) \cdot g(x)$ where $f(x)$ and $g(x)$ are given.

Exercise:

Problem: $f(x) = \frac{6x^2-12x}{x^2+7x-18}$ $g(x) = \frac{x^2-81}{3x^2-27x}$

Solution:

$R(x) = 2$

Exercise:

Problem: $f(x) = \frac{x^2-2x}{x^2+6x-16}$ $g(x) = \frac{x^2-64}{x^2-8x}$

Exercise:

Problem: $f(x) = \frac{4x}{x^2-3x-10}$ $g(x) = \frac{x^2-25}{8x^2}$

Solution:

$R(x) = \frac{x+5}{2x(x+2)}$

Exercise:

Problem: $f(x) = \frac{2x^2+8x}{x^2-9x+20}$ $g(x) = \frac{x-5}{x^2}$

For the following exercises, find $R(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are given.

Exercise:

Problem: $f(x) = \frac{27x^2}{3x-21}$
$g(x) = \frac{3x^2+18x}{x^2+13x+42}$

Solution:

$R(x) = \frac{3x(x+7)}{x-7}$

Exercise:

Problem: $f(x) = \frac{24x^2}{2x-8}$
$g(x) = \frac{4x^3+28x^2}{x^2+11x+28}$

Exercise:

Problem: $f(x) = \frac{16x^2}{4x+36}$
$g(x) = \frac{4x^2-24x}{x^2+4x-45}$

Solution:

$R(x) = \frac{x(x-5)}{x-6}$

Exercise:

Problem: $f(x) = \frac{24x^2}{2x-4}$
$g(x) = \frac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

Exercise:

Problem:

Explain how you find the values of *x* for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

Solution:

Answers will vary.

Exercise:

Problem: Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

Exercise:

Problem:

ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.
ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.
ⓒ Evaluate your answer to part ⓑ when $n = 7$. Did you get the same answer you got in part ⓐ? Why or why not?

Solution:

Answers will vary.

Exercise:

Problem:

ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.
ⓑ Divide $\frac{x^2-1}{x} \div (x+1)$ and explain all your steps.
ⓒ Evaluate your answer to part ⓑ when $x = 5$. Did you get the same answer you got in part ⓐ? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R\left(x\right) = \frac{p(x)}{q(x)}$ where $p\left(x\right)$ and $q\left(x\right)$ are polynomial functions and $q\left(x\right)$ is not zero.

Add and Subtract Rational Expressions
By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Note:
Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review [link].
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
   If you missed this problem, review [link].
3. Subtract: $6\left(2x+1\right)-4\left(x-5\right)$.
   If you missed this problem, review [link].

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Note:
Rational Expression Addition and Subtraction
If p, q, and r are polynomials where $r\ne 0$, then
Equation:

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}\quad\text{and}\quad\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of *x* should be excluded in the next example?

Example:
Exercise:

Problem: Add: $\frac{11x+28}{x+4} + \frac{x^2}{x+4}$.

Solution:

Since the denominator is $x + 4$, we must exclude the value $x = -4$.

$$\frac{11x+28}{x+4} + \frac{x^2}{x+4}, \quad x \neq -4$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{11x+28+x^2}{x+4}$$

Write the degrees in descending order.

$$\frac{x^2+11x+28}{x+4}$$

Factor the numerator.

$$\frac{(x+4)(x+7)}{x+4}$$

Simplify by removing common factors.

$$\frac{\cancel{(x+4)}\,(x+7)}{\cancel{x+4}}$$

Simplify.

$$x + 7$$

The expression simplifies to $x + 7$ but the original expression had a denominator of $x + 4$ so $x \neq -4$.

Note:
Exercise:

Problem: Simplify: $\frac{9x+14}{x+7} + \frac{x^2}{x+7}$.

Solution:

$x + 2$

Note:
Exercise:

Problem: Simplify: $\frac{x^2+8x}{x+5} + \frac{15}{x+5}$.

Solution:

$x + 3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

Example:
Exercise:

Problem: Subtract: $\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$.

Solution:

	$\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-\left(4x^2+x-9\right)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
	$\frac{(x-2)}{(x+3)}$

Note:
Exercise:

Problem: Subtract: $\frac{4x^2-11x+8}{x^2-3x+2} - \frac{3x^2+x-3}{x^2-3x+2}$.

Solution:

$\frac{x-11}{x-2}$

Note:
Exercise:

Problem: Subtract: $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$.

Solution:

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d} + \frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d} + \frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d} + \frac{-5}{d}$
Simplify.	

	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Example:
Exercise:

Problem: Subtract: $\frac{m^2-6m}{m^2-1} - \frac{3m+2}{1-m^2}$.

Solution:

	$\frac{m^2-6m}{m^2-1} - \frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1} - \frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1} - \frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$

Simplify by removing common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Note:
Exercise:

Problem: Subtract: $\frac{y^2-5y}{y^2-4} - \frac{6y-6}{4-y^2}$.

Solution:

$\frac{y+3}{y+2}$

Note:
Exercise:

Problem: Subtract: $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$.

Solution:

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12} + \frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied

the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$

$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$

$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{array}{rl} 12 = & 2\cdot 2\cdot 3 \\ 18 = & 2\cdot \,3\cdot 3 \\ \hline \text{LCD} = & 2\cdot 2\cdot 3\cdot 3 \\ \text{LCD} = & 36 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Note:
Find the least common denominator of rational expressions.

Factor each denominator completely.
List the factors of each denominator. Match factors vertically when possible.
Bring down the columns by including all factors, but do not include common factors twice.
Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

Example:
Exercise:

Problem:

ⓐ Find the LCD for the expressions $\frac{8}{x^2-2x-3}$, $\frac{3x}{x^2+4x+3}$ and ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Solution:

ⓐ

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$x^2-2x-3=(x+1)(x-3)$ $x^2+4x+3=(x+1)(x+3)$ LCD $=(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

ⓑ

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Note:
Exercise:

Problem:

ⓐ Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Solution:

ⓐ $(x-4)(x+3)(x+4)$
ⓑ $\frac{2x+8}{(x-4)(x+3)(x+4)}$,
$\frac{x+3}{(x-4)(x+3)(x+4)}$

Note:
Exercise:

Problem:

ⓐ Find the LCD for the expressions $\frac{3x}{x^2-3x+10}$, $\frac{5}{x^2+3x+2}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Solution:

ⓐ $(x+2)(x-5)(x+1)$
ⓑ $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$,
$\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example:
How to Add Rational Expressions with Unlike Denominators
Exercise:

Problem: Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution:

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$.	$x-3:(x-3)$ $x-2\quad:\quad(x-2)$ LCD : $(x-3)(x-2)$
		$\frac{3}{x-3}+\frac{2}{x-2}$
	Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$.	$\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$
	Keep the denominators factored!	$\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Note:
Exercise:

Problem: Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Solution:

$\frac{7x-4}{(x-2)(x+3)}$

Note:
Exercise:

Problem: Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Solution:

$\frac{7m+25}{(m+3)(m+4)}$

The steps used to add rational expressions are summarized here.

Note:
Add or subtract rational expressions.

Determine if the expressions have a common denominator.

- **Yes –** go to step 2.
- **No –** Rewrite each rational expression with the LCD.
  - Find the LCD.
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Add or subtract the rational expressions.
Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

Example:
Exercise:

Problem: Add: $\frac{8}{x^2-2x-3} + \frac{3x}{x^2+4x+3}$.

Solution:

	$\frac{8}{x^2-2x-3} + \frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	

Find the LCD. $\begin{array}{l} x^2-2x-3=(x+1)(x-3) \\ x^2+4x+3=(x+1)\quad(x+3) \\ \hline \text{LCD}=(x+1)(x-3)(x+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Note:
Exercise:

Problem: Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

Solution:

$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$

Note:
Exercise:

Problem: Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Solution:

$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example:
Exercise:

Problem: Subtract: $\frac{8y}{y^2-16} - \frac{4}{y-4}$.

Solution:

	$\frac{8y}{y^2-16} - \frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} y^2-16 &= (y-4)(y+4) \\ \underline{y-4} &\underline{= y-4} \\ \text{LCD} &= (y-4)(y+4) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)} - \frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)} - \frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	

	$\frac{8y - 4y - 16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y - 16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Note:
Exercise:

Problem: Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

Solution:

$\frac{1}{x-2}$

Note:
Exercise:

Problem: Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

Solution:

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example:
Exercise:

Problem: Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Solution:

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{rl} n^2+n-6 &= (n-2)(n+3) \\ n-2 &= (n-2) \\ \hline \text{LCD} &= (n-2)(n+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$

Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

Note:
Exercise:

Problem: Subtract : $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Solution:

$\frac{5x+1}{(x-6)(x+1)}$

Note:
Exercise:

Problem: Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Solution:

$\frac{y+3}{y+4}$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example:
Exercise:

Problem: Subtract: $\frac{4}{a^2+6a+5} - \frac{3}{a^2+7a+10}$.

Solution:

	$\frac{4}{a^2+6a+5} - \frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)} - \frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{r} a^2+6a+5 \quad = (a+1)(a+5) \\ a^2+7a+10 \;= \qquad\qquad (a+5)(a+2) \\ \hline \text{LCD} = (a+1)(a+5)(a+2) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)} - \frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)} - \frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$

Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

Note:
Exercise:

Problem: Subtract: $\frac{3}{b^2-4b-5} - \frac{2}{b^2-6b+5}$.

Solution:

$\frac{1}{(b+1)(b-1)}$

Note:
Exercise:

Problem: Subtract: $\frac{4}{x^2-4} - \frac{3}{x^2-x-2}$.

Solution:

$\frac{1}{(x+2)(x+1)}$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

Example:
Exercise:

Problem: Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution:

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{rl} u-1 &= (u-1) \\ u &= u \\ u^2-u &= u(u-1) \\ \hline \text{LCD} &= u(u-1) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u}+\frac{1 \cdot (u-1)}{u \cdot (u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u \cdot (u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

Note:
Exercise:

Problem: Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

Solution:

$\frac{v+3}{v+1}$

Note:
Exercise:

Problem: Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

Solution:

$\frac{3w}{w+7}$

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

Example:
Exercise:

Problem: Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+5}{x-2}$ and $g(x) = \frac{5x+18}{x^2-4}$.

Solution:

	$R(x) = f(x) - g(x)$
Substitute in the functions $f(x)$,$g(x)$.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{x^2-4}$

Factor the denominators.	$R(x)=\frac{x+5}{x-2}-\frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} x-2 &= (x-2) \\ x^2-4 &= (x-2)(x+2) \\ \hline \text{LCD} &= (x-2)(x+2) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x)=\frac{(x+5)(x+2)}{(x-2)(x+2)}-\frac{5x+18}{(x-2)(x+2)}$
Write as one rational expression.	$R(x)=\frac{(x+5)(x+2)-(5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x)=\frac{x^2+7x+10-5x-18}{(x-2)(x+2)}$
	$R(x)=\frac{x^2+2x-8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x)=\frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$
Simplify.	$R(x)=\frac{(x+4)}{(x+2)}$

Note:
Exercise:

Problem: Find $R(x)=f(x)-g(x)$ where $f(x)=\frac{x+1}{x+3}$ and $g(x)=\frac{x+17}{x^2-x-12}$.

Solution:

$\frac{x-7}{x-4}$

Note:
Exercise:

Problem: Find $R(x) = f(x) + g(x)$ where $f(x) = \frac{x-4}{x+3}$ and $g(x) = \frac{4x+6}{x^2-9}$.

Solution:

$\frac{x^2-3x+18}{(x+3)(x-3)}$

Note:
Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r \neq 0$, then
  $$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \text{ and } \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**

  Factor each expression completely.
  List the factors of each expression. Match factors vertically when possible.
  Bring down the columns.
  Write the LCD as the product of the factors.

- **How to add or subtract rational expressions.**

  Determine if the expressions have a common denominator.
  - Yes – go to step 2.
  - No – Rewrite each rational expression with the LCD.
    - Find the LCD.
    - Rewrite each rational expression as an equivalent rational expression with the LCD.

Add or subtract the rational expressions.
Simplify, if possible.

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

Exercise:

Problem: $\frac{2}{15} + \frac{7}{15}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\frac{7}{24} + \frac{11}{24}$

Exercise:

Problem: $\frac{3c}{4c-5} + \frac{5}{4c-5}$

Solution:

$\frac{3c+5}{4c-5}$

Exercise:

Problem: $\frac{7m}{2m+n} + \frac{4}{2m+n}$

Exercise:

Problem: $\frac{2r^2}{2r-1} + \frac{15r-8}{2r-1}$

Solution:

$r + 8$

Exercise:

Problem: $\frac{3s^2}{3s-2} + \frac{13s-10}{3s-2}$

Exercise:

Problem: $\frac{2w^2}{w^2-16} + \frac{8w}{w^2-16}$

Solution:

$\frac{2w}{w-4}$

Exercise:

Problem: $\frac{7x^2}{x^2-9} + \frac{21x}{x^2-9}$

In the following exercises, subtract.

Exercise:

Problem: $\frac{9a^2}{3a-7} - \frac{49}{3a-7}$

Solution:

$3a + 7$

Exercise:

Problem: $\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

Exercise:

Problem: $\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Solution:

$\frac{m-2}{2}$

Exercise:

Problem: $\frac{2n^2}{4n-32} - \frac{18n-16}{4n-32}$

Exercise:

Problem: $\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Solution:

$\frac{p+3}{p+5}$

Exercise:

Problem: $\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

Exercise:

Problem: $\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2+5r+30}{r^2-49}$

Solution:

$\frac{r+9}{r+7}$

Exercise:

Problem: $\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

Exercise:

Problem: $\frac{10v}{2v-1} + \frac{2v+4}{1-2v}$

Solution:

4

Exercise:

Problem: $\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

Exercise:

Problem: $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Solution:

$x+2$

Exercise:

Problem: $\frac{6y^2+2y-11}{3y-7} + \frac{3y^2-3y+17}{7-3y}$

Exercise:

Problem: $\frac{z^2+6z}{z^2-25} - \frac{3z+20}{25-z^2}$

Solution:

$\frac{z+4}{z-5}$

Exercise:

Problem: $\frac{a^2+3a}{a^2-9} - \frac{3a-27}{9-a^2}$

Exercise:

Problem: $\frac{2b^2+30b-13}{b^2-49} - \frac{2b^2-5b-8}{49-b^2}$

Solution:

$\frac{4b-3}{b-7}$

Exercise:

Problem: $\frac{c^2+5c-10}{c^2-16} - \frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, ⓐ find the LCD for the given rational expressions ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Exercise:

Problem: $\frac{5}{x^2-2x-8}$, $\frac{2x}{x^2-x-12}$

Solution:

ⓐ $(x+2)(x-4)(x+3)$
ⓑ $\frac{5x+15}{(x+2)(x-4)(x+3)}$,
$\frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

Exercise:

Problem: $\frac{8}{y^2+12y+35}$, $\frac{3y}{y^2+y-42}$

Exercise:

Problem: $\frac{9}{z^2+2z-8}$, $\frac{4z}{z^2-4}$

Solution:

ⓐ $(z-2)(z+4)(z-4)$
ⓑ $\frac{9z-36}{(z-2)(z+4)(z-4)}$, $\frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

Exercise:

Problem: $\frac{6}{a^2+14a+45}$, $\frac{5a}{a^2-81}$

Exercise:

Problem: $\frac{4}{b^2+6b+9}$, $\frac{2b}{b^2-2b-15}$

Solution:

ⓐ $(b+3)(b+3)(b-5)$
ⓑ $\frac{4b-20}{(b+3)(b+3)(b-5)}$, $\frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

Exercise:

Problem: $\frac{5}{c^2-4c+4}$, $\frac{3c}{c^2-7c+10}$

Exercise:

Problem: $\frac{2}{3d^2+14d-5}$, $\frac{5d}{3d^2-19d+6}$

Solution:

ⓐ $(d+5)(3d-1)(d-6)$
ⓑ $\frac{2d-12}{(d+5)(3d-1)(d-6)}$, $\frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

Exercise:

Problem: $\frac{3}{5m^2-3m-2}$, $\frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

Exercise:

Problem: $\frac{7}{10x^2y} + \frac{4}{15xy^2}$

Solution:

$\frac{21y+8x}{30x^2y^2}$

Exercise:

Problem: $\frac{1}{12a^3b^2} + \frac{5}{9a^2b^3}$

Exercise:

Problem: $\frac{3}{r+4} + \frac{2}{r-5}$

Solution:

$\frac{5r-7}{(r+4)(r-5)}$

Exercise:

Problem: $\frac{4}{s-7} + \frac{5}{s+3}$

Exercise:

Problem: $\frac{5}{3w-2} + \frac{2}{w+1}$

Solution:

$\frac{11w+1}{(3w-2)(w+1)}$

Exercise:

Problem: $\frac{4}{2x+5} + \frac{2}{x-1}$

Exercise:

Problem: $\frac{2y}{y+3} + \frac{3}{y-1}$

Solution:

$\frac{2y^2+y+9}{(y+3)(y-1)}$

Exercise:

Problem: $\frac{3z}{z-2} + \frac{1}{z+5}$

Exercise:

Problem: $\frac{5b}{a^2b-2a^2} + \frac{2b}{b^2-4}$

Solution:

$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$

Exercise:

Problem: $\frac{4}{cd+3c} + \frac{1}{d^2-9}$

Exercise:

Problem: $\frac{-3m}{3m-3} + \frac{5m}{m^2+3m-4}$

Solution:

$-\frac{m}{m+4}$

Exercise:

Problem: $\frac{8}{4n+4} + \frac{6}{n^2-n-2}$

Exercise:

Problem: $\frac{3r}{r^2+7r+6} + \frac{9}{r^2+4r+3}$

Solution:

$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$

Exercise:

Problem: $\frac{2s}{s^2+2s-8} + \frac{4}{s^2+3s-10}$

Exercise:

Problem: $\frac{t}{t-6} - \frac{t-2}{t+6}$

Solution:

$\frac{2(7t-6)}{(t-6)(t+6)}$

Exercise:

Problem: $\frac{x-3}{x+6} - \frac{x}{x+3}$

Exercise:

Problem: $\frac{5a}{a+3} - \frac{a+2}{a+6}$

Solution:

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

Exercise:

Problem: $\frac{3b}{b-2} - \frac{b-6}{b-8}$

Exercise:

Problem: $\frac{6}{m+6} - \frac{12m}{m^2-36}$

Solution:

$\frac{-6}{m-6}$

Exercise:

Problem: $\frac{4}{n+4} - \frac{8n}{n^2-16}$

Exercise:

Problem: $\frac{-9p-17}{p^2-4p-21} - \frac{p+1}{7-p}$

Solution:

$\frac{p+2}{p+3}$

Exercise:

Problem: $\frac{-13q-8}{q^2+2q-24} - \frac{q+2}{4-q}$

Exercise:

Problem: $\frac{-2r-16}{r^2+6r-16} - \frac{5}{2-r}$

Solution:

$\frac{3}{r-2}$

Exercise:

Problem: $\frac{2t-30}{t^2+6t-27} - \frac{2}{3-t}$

Exercise:

Problem: $\frac{2x+7}{10x-1}+3$

Solution:

$\frac{4(8x+1)}{10x-1}$

Exercise:

Problem: $\frac{8y-4}{5y+2}-6$

Exercise:

Problem: $\frac{3}{x^2-3x-4}-\frac{2}{x^2-5x+4}$

Solution:

$\frac{x-5}{(x-4)(x+1)(x-1)}$

Exercise:

Problem: $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

Exercise:

Problem: $\frac{5}{x^2+8x-9}-\frac{4}{x^2+10x+9}$

Solution:

$\frac{1}{(x-1)(x+1)}$

Exercise:

Problem: $\frac{3}{2x^2+5x+2}-\frac{1}{2x^2+3x+1}$

Exercise:

Problem: $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

Solution:

$\frac{5a^2+7a-36}{a(a-2)}$

Exercise:

Problem: $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

Exercise:

Problem: $\frac{c}{c+2}+\frac{5}{c-2}-\frac{10c}{c^2-4}$

Solution:

$\frac{c-5}{c+2}$

Exercise:

Problem: $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

Exercise:

Problem: $\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

Solution:

$\frac{3(d+1)}{d+2}$

Exercise:

Problem: $\frac{2q}{q+5}+\frac{3}{q-3}-\frac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find ⓐ $R\left(x\right)=f\left(x\right)+g\left(x\right)$ ⓑ $R\left(x\right)=f\left(x\right)-g\left(x\right)$.

Exercise:

Problem: $f\left(x\right)=\frac{-5x-5}{x^2+x-6}$ and $g\left(x\right)=\frac{x+1}{2-x}$

Solution:

ⓐ $R\left(x\right)=-\frac{(x+8)(x+1)}{(x-2)(x+3)}$ ⓑ $R\left(x\right)=\frac{x+1}{x+3}$

Exercise:

Problem: $f\left(x\right)=\frac{-4x-24}{x^2+x-30}$ and $g\left(x\right)=\frac{x+7}{5-x}$

Exercise:

Problem: $f(x) = \frac{6x}{x^2-64}$ and $g(x) = \frac{3}{x-8}$

Solution:

ⓐ $\frac{3(3x+8)}{(x-8)(x+8)}$
ⓑ $R(x) = \frac{3}{x+8}$

Exercise:

Problem: $f(x) = \frac{5}{x+7}$ and $g(x) = \frac{10x}{x^2-49}$

Writing Exercises

Exercise:

Problem: Donald thinks that $\frac{3}{x} + \frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

Solution:

Answers will vary.

Exercise:

Problem:

Explain how you find the Least Common Denominator of $x^2 + 5x + 4$ and $x^2 - 16$.

Exercise:

Problem: Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x+y}$.
ⓐ Choose numerical values for *x* and *y* and evaluate $\frac{1}{x} + \frac{1}{y}$.
ⓑ Evaluate $\frac{2}{x+y}$ for the same values of *x* and *y* you used in part ⓐ.
ⓒ Explain why Felipe is wrong.
ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

Solution:

ⓐ Answers will vary.
ⓑ Answers will vary.
ⓒ Answers will vary.
ⓓ $\frac{x+y}{xy}$

Exercise:

Problem: Simplify the expression $\frac{4}{n^2+6n+9} - \frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Simplify Complex Rational Expressions
By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review [link].
3. Solve: $\frac{1}{2x}+\frac{1}{4}=\frac{1}{8}$.
   If you missed this problem, review [link].

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:
Equation:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Note:
Complex Rational Expression
A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:
Equation:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.
Equation:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:
Equation:

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$.

Solution:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Rewrite the complex fraction as division. $\frac{6}{x-4} \div \frac{3}{x^2-16}$

Rewrite as the product of first times the reciprocal of the second. $\frac{6}{x-4} \cdot \frac{x^2-16}{3}$

Factor. $\frac{3 \cdot 2}{x-4} \cdot \frac{(x-4)(x+4)}{3}$

Multiply. $\frac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$

Remove common factors. $\frac{\cancel{3} \cdot 2\, \cancel{(x-4)}\, (x+4)}{\cancel{3}\, \cancel{(x-4)}}$

Simplify. $2\,(x+4)$

Are there any value(s) of *x* that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

Solution:

$\frac{2}{3(x-1)}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Solution:

$\frac{1}{2(x-3)}$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution:

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$\dfrac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$\dfrac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

Solution:

$\frac{14}{11}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

Solution:

$\frac{10}{23}$

We follow the same procedure when the complex rational expression contains variables.

Example:
How to Simplify a Complex Rational Expression using Division
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution:

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
		$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Find a common denominator and subtract the fractions in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\dfrac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\dfrac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\dfrac{1}{x-y}$

Note:

Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

Solution:

$\frac{y+x}{y-x}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

Solution:

$\frac{ab}{b-a}$

We summarize the steps here.

Note:
Simplify a complex rational expression by writing it as division.

Simplify the numerator and denominator.
Rewrite the complex rational expression as a division problem.
Divide the expressions.

Example:

Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution:

	$\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$\dfrac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator.	$\dfrac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$
Simplify. (We now have one rational expression over one rational expression.)	$\dfrac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$

Rewrite as fraction division.	$\frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\frac{(n+1)(n-5)}{2}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$.

Solution:

$\frac{b(b+2)(b-5)}{3b-5}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by writing it as division: $\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$.

Solution:

$\frac{3}{c+3}$

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in [link]. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

Example:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution:

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\dfrac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\dfrac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\dfrac{2+1}{3-2}$
	$\dfrac{3}{1}$
	3

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

Solution:

$\frac{7}{3}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

Solution:

$\frac{10}{3}$

We will use the same example as in [link]. Decide which method works better for you.

Example:
How to Simplify a Complex Rational Expressing using the LCD
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution:

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is *xy*.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$

Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by *xy*.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Step 3. Simplify the expression.	Distribute.	$\dfrac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$
		$\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$\dfrac{1}{x-y}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

Solution:

$\frac{b+a}{a^2+b^2}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

Solution:

$\frac{y-x}{xy}$

Note:
Simplify a complex rational expression by using the LCD.

Find the LCD of all fractions in the complex rational expression.
Multiply the numerator and denominator by the LCD.
Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution:

	$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the	

complex rational expression. The LCD is $x^2 - 36 = (x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6} - \frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	
Distribute in the denominator.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right) - (x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right) - \cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Factor the denominator.	$\dfrac{2(x-6)}{4(x+5)}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$
Simplify.	$\dfrac{x-6}{2(x+5)}$

Notice that there are no more factors common to the numerator and denominator.	

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

Solution:

$\frac{3(x-2)}{5x+7}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Solution:

$\frac{x+21}{6x-43}$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Solution:

	$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	

	$\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$\frac{4}{3m-12-2m+6}$
Combine like terms.	$\frac{4}{m-6}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

Solution:

$\frac{3}{5x+22}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

Solution:

$\frac{2\left(2y^2+13y+5\right)}{3y}$

Example:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution:

	$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is $(y+1)(y-1)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$

Simplify the denominator and leave the numerator factored.	$\frac{y(y-1)}{y^2-1+y+1}$
	$\frac{y(y-1)}{y^2+y}$
Factor the denominator and remove factors common with the numerator.	$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\frac{y-1}{y+1}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

Solution:

$\frac{x}{x+4}$

Note:
Exercise:

Problem:

Simplify the complex rational expression by using the LCD: $\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

Solution:

$\frac{x(x+1)}{3(x-1)}$

Note:
Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**

  Simplify the numerator and denominator.
  Rewrite the complex rational expression as a division problem.
  Divide the expressions.

- **How to simplify a complex rational expression by using the LCD.**

  Find the LCD of all fractions in the complex rational expression.
  Multiply the numerator and denominator by the LCD.
  Simplify the expression.

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.
Exercise:

Problem: $\frac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

Solution:

$\frac{a-4}{2a}$

Exercise:

Problem: $\frac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

Exercise:

Problem: $\frac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

Solution:

$\frac{1}{2(c-2)}$

Exercise:

Problem: $\frac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

Exercise:

Problem: $\frac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Solution:

$\frac{12}{13}$

Exercise:

Problem: $\frac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

Exercise:

Problem: $\frac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Solution:

$\frac{20}{57}$

Exercise:

Problem: $\frac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

Exercise:

Problem: $\frac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Solution:

$\frac{n^2+m}{m-n^2}$

Exercise:

Problem: $\frac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

Exercise:

Problem: $\frac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Solution:

$\frac{rt}{t-r}$

Exercise:

Problem: $\frac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

Exercise:

Problem: $\frac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Solution:

$\frac{(x+1)(x-3)}{2}$

Exercise:

Problem: $\frac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

Exercise:

Problem: $\frac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

Solution:

$\frac{4}{a+1}$

Exercise:

Problem: $\frac{4+\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

Exercise:

Problem: $\frac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

Solution:

$\frac{11}{8}$

Exercise:

Problem: $\frac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

Exercise:

Problem: $\frac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

Solution:

19

Exercise:

Problem: $\frac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

Exercise:

Problem: $\frac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

Solution:

$\frac{c^2+c}{c-d^2}$

Exercise:

Problem: $\frac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Exercise:

Problem: $\frac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

Solution:

$\frac{pq}{q-p}$

Exercise:

Problem: $\frac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Exercise:

Problem: $\frac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

Solution:

$\frac{2x-10}{3x+16}$

Exercise:

Problem: $\frac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

Exercise:

Problem: $\frac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

Solution:

$\frac{3z-19}{3z+8}$

Exercise:

Problem: $\frac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

Exercise:

Problem: $\frac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

Solution:

$$\frac{4}{3a-7}$$

Exercise:

Problem: $\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$

Exercise:

Problem: $\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$

Solution:

$$\frac{2c+29}{5c}$$

Exercise:

Problem: $\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

Exercise:

Problem: $\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

Solution:

$$\frac{2p-5}{5}$$

Exercise:

Problem: $\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

Exercise:

Problem: $\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

Solution:

$$\frac{m(m-5)}{(4m-19)(m+5)}$$

Exercise:

Problem: $\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

In the following exercises, simplify each complex rational expression using either method.

Exercise:

Problem: $\frac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

Solution:

$$\frac{13}{24}$$

Exercise:

Problem: $\frac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

Exercise:

Problem: $\frac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

Solution:

$$2(a-4)$$

Exercise:

Problem: $\frac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

Exercise:

Problem: $\frac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

Solution:

$\frac{3mn}{n-m}$

Exercise:

Problem: $\frac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

Exercise:

Problem: $\frac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

Solution:

$\frac{(x-1)(x-2)}{6}$

Exercise:

Problem: $\frac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Writing Exercises

Exercise:

Problem:

In this section, you learned to simplify the complex fraction $\frac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Solution:

Answers will vary.

Exercise:

Problem:

Efraim wants to start simplifying the complex fraction $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\frac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

Solve Rational Equations
By the end of this section, you will be able to:

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Note:
Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$.
   If you missed this problem, review [link].
2. Solve: $n^2 - 5n - 36 = 0$.
   If you missed this problem, review [link].
3. Solve the formula $5x + 2y = 10$ for y.
   If you missed this problem, review [link].

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will *solve* a **rational equation**.

Note:
Rational Equation
A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.
Equation:

$$\begin{array}{ccc} \text{Rational Expression} & & \text{Rational Equation} \\ \frac{1}{8}x + \frac{1}{2} & & \frac{1}{8}x + \frac{1}{2} = \frac{1}{4} \\ \frac{y+6}{y^2-36} & & \frac{y+6}{y^2-36} = y + 1 \\ \frac{1}{n-3} + \frac{1}{n+4} & & \frac{1}{n-3} + \frac{1}{n+4} = \frac{15}{n^2+n-12} \end{array}$$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the

fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Note:
Extraneous Solution to a Rational Equation
An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example:
How to Solve a Rational Equation
Exercise:

Problem: Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Solution:

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$ $6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$ $6+2x=5x$
Step 4. Solve the resulting equation.	Simplify.	$6=3x$ $2=x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x=2$ into the original equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$ $\frac{1}{2}+\frac{1}{3}\overset{?}{=}\frac{5}{6}$ $\frac{3}{6}+\frac{2}{6}\overset{?}{=}\frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}\checkmark$ The solution is $x=2$.

Note:
Exercise:

Problem: Solve: $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

Solution:

$y=-\frac{7}{15}$

Note:
Exercise:

Problem: Solve: $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

Solution:

$x = \frac{13}{15}$

The steps of this method are shown.

Note:
Solve equations with rational expressions.

Note any value of the variable that would make any denominator zero.
Find the least common denominator of*all*denominators in the equation.
Clear the fractions by multiplying both sides of the equation by the LCD.
Solve the resulting equation.
Check:
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example:
How to Solve a Rational Equation using the Zero Product Property
Exercise:

Problem: Solve: $1 - \frac{5}{y} = -\frac{6}{y^2}$.

Solution:

	$1 - \frac{5}{y} = -\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1 - \frac{5}{y} = -\frac{6}{y^2}, y \neq 0$
Find the least common denominator of all	

denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$
Check. We did not get 0 as an algebraic solution. Check $y=2$ and $y=3$ in the original equation. $1-\frac{5}{y}=-\frac{6}{y^2}$ $1-\frac{5}{y}=-\frac{6}{y^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2}$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $\frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4}$ $-\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{3}{2}=-\frac{3}{2}$ ✓ $-\frac{2}{3}=-\frac{2}{3}$ ✓	

	The solution is $y = 2$, $y = 3$.

Note:
Exercise:

Problem: Solve: $1 - \frac{2}{x} = \frac{15}{x^2}$.

Solution:

$x = -3, x = 5$

Note:
Exercise:

Problem: Solve: $1 - \frac{4}{y} = \frac{12}{y^2}$.

Solution:

$y = -2, y = 6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example:
Exercise:

Problem: Solve: $\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$.

Solution:

	$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$
Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Distribute.	$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Remove common factors.	$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\cancel{(x-2)}\frac{4}{\cancel{x-2}}=\cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$
Simplify.	$2(x-2)+4(x+2)=x-1$
Distribute.	$2x-4+4x+8=x-1$
Solve.	$6x+4=x-1$ $5x=-5$ $x=-1$
Check: We did not get 2 or −2 as algebraic solutions.	

Check $x = -1$ in the original equation. $$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$$ $$\frac{2}{(-1)+2} + \frac{4}{(-1)-2} \stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4}$$ $$\frac{2}{1} + \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$$ $$\frac{6}{3} - \frac{4}{3} \stackrel{?}{=} \frac{2}{3}$$ $$\frac{2}{3} = \frac{2}{3} \checkmark$$	
	The solution is $x = -1$.

Note:
Exercise:

Problem: Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$.

Solution:

$x = \frac{2}{3}$

Note:
Exercise:

Problem: Solve: $\frac{5}{y+3} + \frac{2}{y-3} = \frac{5}{y^2-9}$.

Solution:

$y = 2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example:

Exercise:

Problem: Solve: $\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$.

Solution:

	$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}=\frac{3}{m-1}$
Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.	$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m+11=5(m-1)-3(m-4)$
Solve the resulting equation.	$m+11=5m-5-3m+12$ $4=m$
Check. The only algebraic solution was 4, but we said that 4 would	

make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

Note:
Exercise:

Problem: Solve: $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

Solution:

There is no solution.

Note:
Exercise:

Problem: Solve: $\frac{y-6}{y^2+3y-4} = \frac{2}{y+4} + \frac{7}{y-1}$.

Solution:

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

Example:
Exercise:

Problem: Solve: $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$.

Solution:

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$
Find the least common denominator. The LCD is $(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4(y^2-36)$
Solve the resulting equation.	$y^2-6y=72+4y^2-144$ $0=3y^2+6y-72$ $0=3(y^2+2y-24)$ $0=3(y+6)(y-4)$ $y=-6, y=4$
Check.	

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation. $\frac{y}{y+6}=\frac{72}{y^2-36}+4$ $\frac{4}{4+6}\overset{?}{=}\frac{72}{4^2-36}+4$ $\frac{4}{10}\overset{?}{=}\frac{72}{-20}+4$ $\frac{4}{10}\overset{?}{=}-\frac{36}{10}+\frac{40}{10}$ $\frac{4}{10}=\frac{4}{10}$ ✓	
	The solution is $y=4$.

Note:
Exercise:

Problem: Solve: $\frac{x}{x+4}=\frac{32}{x^2-16}+5$.

Solution:

$x=3$

Note:
Exercise:

Problem: Solve: $\frac{y}{y+8}=\frac{128}{y^2-64}+9$.

Solution:

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example:

Exercise:

Problem: Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$.

Solution:

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator. The LCD is $12(x-1)(x+1)$.	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2 - 2x + 9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2 - 2x + 9$
Solve the resulting equation.	$6x^2 + 6x - 8x + 8 = 5x^2 - 2x + 9$ $x^2 - 1 = 0$ $(x-1)(x+1) = 0$

$x = 1$ or $x = -1$

Check.

$x = 1$ and $x = -1$ are extraneous solutions.

	The equation has no solution.

Note:
Exercise:

Problem: Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

Solution:

There is no solution.

Note:
Exercise:

Problem: Solve: $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-16}{8z^2+2z-64}$.

Solution:

There is no solution.

Example:
Exercise:

Problem: Solve: $\frac{4}{3x^2-10x+3} + \frac{3}{3x^2+2x-1} = \frac{2}{x^2-2x-3}$.

Solution:

	$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}$ $x \neq -1, x \neq \frac{1}{3}, x \neq 3$
Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.	
Clear the fractions.	
$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$	
Simplify.	$4(x+1)+3(x-3)=2(3x-1)$
Distribute.	$4x+4+3x-9=6x-2$
Simplify.	$7x-5=6x-2$
	$x=3$
The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

Note:
Exercise:

Problem: Solve: $\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$.

Solution:

There is no solution.

Note:
Exercise:

Problem: Solve: $\frac{5}{x^2+2x-3} - \frac{3}{x^2+x-2} = \frac{1}{x^2+5x+6}$.

Solution:

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example:
Exercise:

Problem:

For rational function, $f(x) = \frac{2x-6}{x^2-8x+15}$, ⓐ find the domain of the function, ⓑ solve $f(x) = 1$, and ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

	$x^2 - 8x + 15 = 0$
Factor the trinomial.	$(x-3)(x-5) = 0$
Use the Zero Product Property.	$x - 3 = 0 \quad x - 5 = 0$
Solve.	$x = 3 \quad x = 5$
	The domain is all real numbers except $x \neq 3, x \neq 5$.

.

ⓑ

	$f(x) = 1$

Substitute in the rational expression.	$\frac{2x-6}{x^2-8x+15}=1$
Factor the denominator.	$\frac{2x-6}{(x-3)(x-5)}=1$
Multiply both sides by the LCD, $(x-3)(x-5)$.	$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$
Simplify.	$2x-6=x^2-8x+15$
Solve.	$0=x^2-10x+21$
Factor.	$0=(x-7)(x-3)$
Use the Zero Product Property.	$x-7=0 \quad x-3=0$
Solve.	$x=7 \quad x=3$

ⓒ The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1), (3,1)$.

Note:
Exercise:

Problem:

For rational function, $f(x)=\frac{8-x}{x^2-7x+12}$, ⓐ find the domain of the function ⓑ solve $f(x)=3$ ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain is all real numbers except $x\neq 3$ and $x\neq 4$. ⓑ $x=2, x=\frac{14}{3}$
ⓒ $(2,3), \left(\frac{14}{3},3\right)$

Note:
Exercise:

Problem:

For rational function, $f(x) = \frac{x-1}{x^2-6x+5}$, ⓐ find the domain of the function ⓑ solve $f(x) = 4$ ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain is all real numbers except $x \neq 1$ and $x \neq 5$. ⓑ $x = \frac{21}{4}$ ⓒ $\left(\frac{21}{4}, 4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

	$m = \frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x - x_1$.	$m(x - x_1) = \left(\frac{y-y_1}{x-x_1}\right)(x - x_1)$
Simplify.	$m(x - x_1) = y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 = m(x - x_1)$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2, 3)$. We will add one more step to solve for y.

Example:
Exercise:

Problem: Solve: $m = \frac{y-2}{x-3}$ for y.

Solution:

	$m = \frac{y-2}{x-3}$
Note any value of the variable that would make any denominator zero.	$m = \frac{y-2}{x-3}, x \neq 3$
Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.	$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$
Simplify.	$xm - 3m = y - 2$
Isolate the term with *y*.	$xm - 3m + 2 = y$

Note:
Exercise:

Problem: Solve: $m = \frac{y-5}{x-4}$ for y.

Solution:

$y = mx - 4m + 5$

Note:
Exercise:

Problem: Solve: $m = \frac{y-1}{x+5}$ for y.

Solution:

$y = mx + 5m + 1$

Remember to multiply both sides by the LCD in the next example.

Example:
Exercise:

Problem: Solve: $\frac{1}{c} + \frac{1}{m} = 1$ for c.

Solution:

	$\frac{1}{c} + \frac{1}{m} = 1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c} + \frac{1}{m} = 1, c \neq 0, m \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, *cm*.	$cm\left(\frac{1}{c} + \frac{1}{m}\right) = cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right) + cm\frac{1}{m} = cm(1)$
Simplify.	$m + c = cm$
Collect the terms with c to the right.	$m = cm - c$
Factor the expression on the right.	$m = c(m - 1)$
To isolate c, divide both sides by $m - 1$.	$\frac{m}{m-1} = \frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1} = c$
Notice that even though we excluded $c = 0, m = 0$ from the original equation, we must also now state that $m \neq 1$.	

Note:
Exercise:

Problem: Solve: $\frac{1}{a} + \frac{1}{b} = c$ for a.

Solution:

$a = \frac{b}{cb-1}$

Note:
Exercise:

Problem: Solve: $\frac{2}{x} + \frac{1}{3} = \frac{1}{y}$ for y.

Solution:

$y = \frac{3x}{x+6}$

Note:
Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

Key Concepts

- **How to solve equations with rational expressions.**

  Note any value of the variable that would make any denominator zero.
  Find the least common denominator of all denominators in the equation.
  Clear the fractions by multiplying both sides of the equation by the LCD.
  Solve the resulting equation.
  Check:
    - If any values found in Step 1 are algebraic solutions, discard them.
    - Check any remaining solutions in the original equation.

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve each rational equation.

Exercise:

Problem: $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Solution:

$a=10$

Exercise:

Problem: $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

Exercise:

Problem: $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Solution:

$v=\frac{40}{21}$

Exercise:

Problem: $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

Exercise:

Problem: $1-\frac{2}{m}=\frac{8}{m^2}$

Solution:

$m=-2, m=4$

Exercise:

Problem: $1+\frac{4}{n}=\frac{21}{n^2}$

Exercise:

Problem: $1+\frac{9}{p}=\frac{-20}{p^2}$

Solution:

$p=-5, p=-4$

Exercise:

Problem: $1 - \frac{7}{q} = \frac{-6}{q^2}$

Exercise:

Problem: $\frac{5}{3v-2} = \frac{7}{4v}$

Solution:

$v = 14$

Exercise:

Problem: $\frac{8}{2w+1} = \frac{3}{w}$

Exercise:

Problem: $\frac{3}{x+4} + \frac{7}{x-4} = \frac{8}{x^2-16}$

Solution:

$x = -\frac{4}{5}$

Exercise:

Problem: $\frac{5}{y-9} + \frac{1}{y+9} = \frac{18}{y^2-81}$

Exercise:

Problem: $\frac{8}{z-10} - \frac{7}{z+10} = \frac{5}{z^2-100}$

Solution:

$z = -145$

Exercise:

Problem: $\frac{9}{a+11} - \frac{6}{a-11} = \frac{6}{a^2-121}$

Exercise:

Problem: $\frac{-10}{q-2} - \frac{7}{q+4} = 1$

Solution:

$q = -18, q = -1$

Exercise:

Problem: $\frac{2}{s+7} - \frac{3}{s-3} = 1$

Exercise:

Problem: $\frac{v-10}{v^2-5v+4} = \frac{3}{v-1} - \frac{6}{v-4}$

Solution:

no solution

Exercise:

Problem: $\frac{w+8}{w^2-11w+28} = \frac{5}{w-7} + \frac{2}{w-4}$

Exercise:

Problem: $\frac{x-10}{x^2+8x+12} = \frac{3}{x+2} + \frac{4}{x+6}$

Solution:

no solution

Exercise:

Problem: $\frac{y-5}{y^2-4y-5} = \frac{1}{y+1} + \frac{1}{y-5}$

Exercise:

Problem: $\frac{b+3}{3b} + \frac{b}{24} = \frac{1}{b}$

Solution:

$b = -8$

Exercise:

Problem: $\frac{c+3}{12c} + \frac{c}{36} = \frac{1}{4c}$

Exercise:

Problem: $\frac{d}{d+3} = \frac{18}{d^2-9} + 4$

Solution:

$d = 2$

Exercise:

Problem: $\frac{m}{m+5} = \frac{50}{m^2-25} + 6$

Exercise:

Problem: $\frac{n}{n+2} - 3 = \frac{8}{n^2-4}$

Solution:

$m = 1$

Exercise:

Problem: $\frac{p}{p+7} - 8 = \frac{98}{p^2-49}$

Exercise:

Problem: $\frac{q}{3q-9} - \frac{3}{4q+12} = \frac{7q^2+6q+63}{24q^2-216}$

Solution:

no solution

Exercise:

Problem: $\frac{r}{3r-15} - \frac{1}{4r+20} = \frac{3r^2+17r+40}{12r^2-300}$

Exercise:

Problem: $\frac{s}{2s+6} - \frac{2}{5s+5} = \frac{5s^2-3s-7}{10s^2+40s+30}$

Solution:

$s = \frac{5}{4}$

Exercise:

Problem: $\frac{t}{6t-12} - \frac{5}{2t+10} = \frac{t^2-23t+70}{12t^2+36t-120}$

Exercise:

Problem: $\frac{2}{x^2+2x-8} - \frac{1}{x^2+9x+20} = \frac{4}{x^2+3x-10}$

Solution:

$x = -\frac{4}{3}$

Exercise:

Problem: $\frac{5}{x^2+4x+3} + \frac{2}{x^2+x-6} = \frac{3}{x^2-x-2}$

Exercise:

Problem: $\frac{3}{x^2-5x-6} + \frac{3}{x^2-7x+6} = \frac{6}{x^2-1}$

Solution:

no solution

Exercise:

Problem: $\frac{2}{x^2+2x-3} + \frac{3}{x^2+4x+3} = \frac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

Exercise:

For rational function, $f(x) = \frac{x-2}{x^2+6x+8}$,
ⓐ find the domain of the function
ⓑ solve $f(x) = 5$
Problem: ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain is all real numbers except $x \neq -2$ and $x \neq -4$.
ⓑ $x = -3, x = -\frac{14}{5}$
ⓒ $(-3, 5)$, $\left(-\frac{14}{5}, 5\right)$

Exercise:

For rational function, $f(x) = \frac{x+1}{x^2-2x-3}$,
ⓐ find the domain of the function
ⓑ solve $f(x) = 1$
Problem: ⓒ find the points on the graph at this function value.

Exercise:

For rational function, $f(x) = \frac{2-x}{x^2-7x+10}$,
ⓐ find the domain of the function
ⓑ solve $f(x) = 2$
Problem: ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain is all real numbers except $x \neq 2$ and $x \neq 5$.
ⓑ $x = \frac{9}{2}$,
ⓒ $\left(\frac{9}{2}, 2\right)$

Exercise:

Problem: For rational function, $f\left(x\right) = \frac{5-x}{x^2+5x+6}$,
ⓐ find the domain of the function
ⓑ solve $f\left(x\right) = 3$
ⓒ the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

Exercise:

Problem: $\frac{C}{r} = 2\pi$ for r.

Solution:

$r = \frac{C}{2\pi}$

Exercise:

Problem: $\frac{I}{r} = P$ for r.

Exercise:

Problem: $\frac{v+3}{w-1} = \frac{1}{2}$ for w.

Solution:

$w = 2v + 7$

Exercise:

Problem: $\frac{x+5}{2-y} = \frac{4}{3}$ for y.

Exercise:

Problem: $a = \frac{b+3}{c-2}$ for c.

Solution:

$c = \frac{b+3+2a}{a}$

Exercise:

Problem: $m=\frac{n}{2-n}$ for n.

Exercise:

Problem: $\frac{1}{p}+\frac{2}{q}=4$ for p.

Solution:

$p=\frac{q}{4q-2}$

Exercise:

Problem: $\frac{3}{s}+\frac{1}{t}=2$ for s.

Exercise:

Problem: $\frac{2}{v}+\frac{1}{5}=\frac{3}{w}$ for w.

Solution:

$w=\frac{15v}{10+v}$

Exercise:

Problem: $\frac{6}{x}+\frac{2}{3}=\frac{1}{y}$ for y.

Exercise:

Problem: $\frac{m+3}{n-2}=\frac{4}{5}$ for n.

Solution:

$n=\frac{5m+23}{4}$

Exercise:

Problem: $r=\frac{s}{3-t}$ for t.

Exercise:

Problem: $\frac{E}{c}=m^2$ for c.

Solution:

$c=\frac{E}{m^2}$

Exercise:

Problem: $\frac{R}{T} = W$ for T.

Exercise:

Problem: $\frac{3}{x} - \frac{5}{y} = \frac{1}{4}$ for y.

Solution:

$y = \frac{20x}{12-x}$

Exercise:

Problem: $c = \frac{2}{a} + \frac{b}{5}$ for a.

Writing Exercises

Exercise:

Problem:

Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Solution:

Answers will vary.

Exercise:

Problem:

Alek thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations involving functions.			
solve rational equations for a specific variable.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

extraneous solution to a rational equation
: An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

rational equation
: A rational equation is an equation that contains a rational expression.

Solve Applications with Rational Equations
By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

Note:
Before you get started, take this readiness quiz.

1. Solve: $2(n-1)-4=-10$.
   If you missed this problem, review [link].
2. An express train and a charter bus leave Chicago to travel to Champaign. The express train can make the trip in two hours and the bus takes five hours for the trip. The speed of the express train is 42 miles per hour faster than the speed of the bus. Find the speed of the bus.
   If you missed this problem, review [link].
3. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
   If you missed this problem, review [link].

Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

Note:
Proportion
A **proportion** is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\neq 0, d\neq 0$.
The proportion is read "*a* is to *b* as *c* is to *d*."

The equation $\frac{1}{2}=\frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2}=\frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

Example:
Exercise:

Problem: Solve: $\frac{n}{n+14}=\frac{5}{7}$.

Solution:

		$\frac{n}{n+14}=\frac{5}{7} \qquad n \neq -14$
Multiply both sides by LCD.		$7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right)$
Remove common factors on each side.		$7n=5(n+14)$
Simplify.		$7n=5n+70$
Solve for *n*.		$2n=70$
		$n=35$
Check.		
	$\frac{n}{n+14}=\frac{5}{7}$	
Substitute $n=35$.	$\frac{35}{35+14}\stackrel{?}{=}\frac{5}{7}$	
Simplify.	$\frac{35}{49}\stackrel{?}{=}\frac{5}{7}$	
Show common factors.	$\frac{5\cdot 7}{7\cdot 7}\stackrel{?}{=}\frac{5}{7}$	
Simplify.	$\frac{5}{7}=\frac{5}{7}\checkmark$	

Note:
Exercise:

Problem: Solve the proportion: $\frac{y}{y+55}=\frac{3}{8}$.

Solution:

$y = 33$

Note:
Exercise:

Problem: Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

Solution:

$z = 14$

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\frac{n}{n+14} = \frac{5}{7} \qquad\qquad \frac{n}{n+14} = \frac{5}{7}$$

$$7(n+14)\left(\frac{n}{n+14}\right) = 7(n+14)\left(\frac{5}{7}\right) \qquad\qquad \frac{n}{n+14} = \frac{5}{7}$$

$$7n = 5(n+14) \qquad\qquad 7n = 5(n+14)$$

For any proportion, $\frac{a}{b} = \frac{c}{d}$, we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\frac{a}{b} = \frac{c}{d} \qquad\qquad \frac{a}{b} = \frac{c}{d}$$

$$bd\left(\frac{a}{b} = \frac{c}{d}\right)bd \qquad\qquad \frac{a}{b} = \frac{c}{d}$$

$$ad = bc \qquad\qquad ad = bc$$

To solve applications with proportions, we will follow our usual strategy for solving applications But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

Example:
Exercise:

Problem:

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child’s weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution:

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion—be careful of the units.	
$\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by the LCD, 400.	$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.	$\cancel{25} \cdot 16\left(\frac{5}{\cancel{25}}\right) = \cancel{80} \cdot 5\left(\frac{a}{\cancel{80}}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.	$16 \cdot 5 = 5a$
Solve for a.	$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
	$16 = a$
Check. Is the answer reasonable? Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense. Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Note:
Exercise:

Problem:

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Solution:

The pediatrician will prescribe 12 ml of acetaminophen to Emilia.

Note:
Exercise:

Problem:

For every 1 kilogram (kg) of a child’s weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Solution:

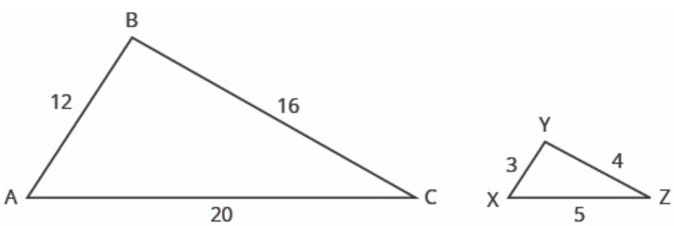
The pediatrician will prescribe 180 mg of fever reducer to Isabella.

Solve similar figure applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

Note:
Similar Figures
Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

For example, the two triangles in [link] are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ.



$m\angle A = m\angle X$

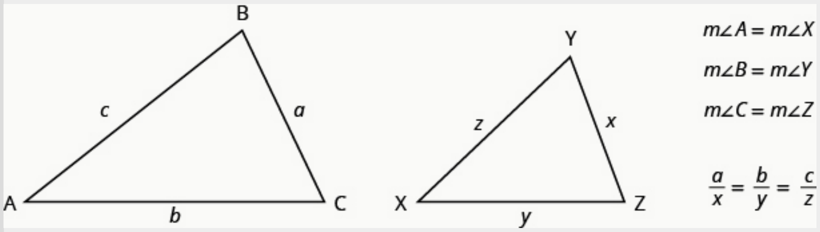
$m\angle B = m\angle Y$

$m\angle C = m\angle Z$

$$\frac{16}{4} = \frac{20}{5} = \frac{12}{3}$$

This is summed up in the Property of Similar Triangles.

Note:
Property of Similar Triangles
If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.

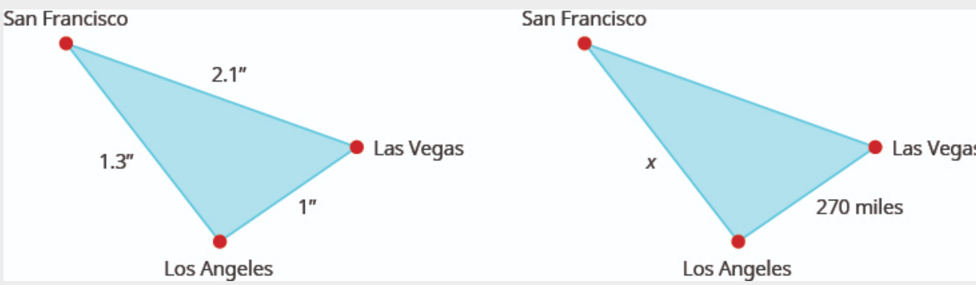


To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

Example:
Exercise:

Problem:

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. If the actual distance from Los Angeles to Las Vegas is 270 miles, find the distance from Los Angeles to San Francisco.



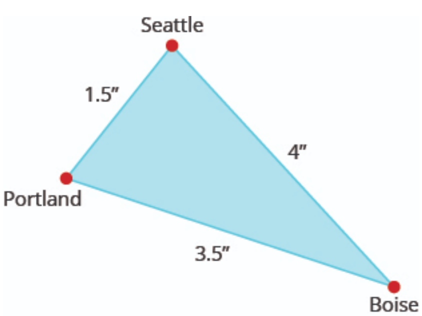
Solution:

Since the triangles are similar, the corresponding sides are proportional.

Read the problem. Draw the figures and label it with the given information.	The figures are shown above.
Identify what we are looking for.	the actual distance from Los Angeles to San Francisco
Name the variables.	Let x = distance from Los Angeles

	to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches".	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
Solve the equation.	$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$
	$x = 351$
Check. On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense. Check $x = 351$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. The actual distance from Seattle to Boise is 400 miles.



Note:
Exercise:

Problem: Find the actual distance from Seattle to Portland.

Solution:

The distance is 150 miles.

Note:
Exercise:

Problem: Find the actual distance from Portland to Boise.

Solution:

The distance is 350 miles.

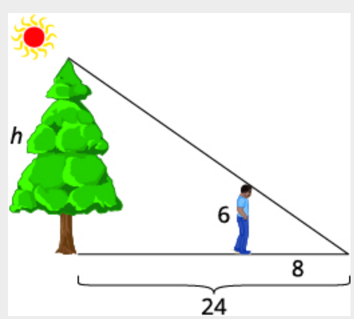
We can use similar figures to find heights that we cannot directly measure.

Example:
Exercise:

Problem:

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution:

Read the problem and draw a figure.	
We are looking for *h*, the height of the tree.	

We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24} = \frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right) = 24\left(\frac{h}{24}\right)$
	$18 = h$
Simplify.	
Check.	

Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow. Check $h = 18$ in the original proportion.

$$\frac{6}{8} = \frac{h}{24}$$

$$\frac{6}{8} \stackrel{?}{=} \frac{18}{24}$$

$$\frac{3}{4} = \frac{3}{4} \checkmark$$

Note:
Exercise:

Problem:

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

Solution:

The telephone pole is 40 feet tall.

Note:
Exercise:

Problem:

A pine tree casts a shadow of 80 feet next to a 30 foot tall building which casts a 40 feet shadow. How tall is the pine tree?

Solution:

The pine tree is 60 feet tall.

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

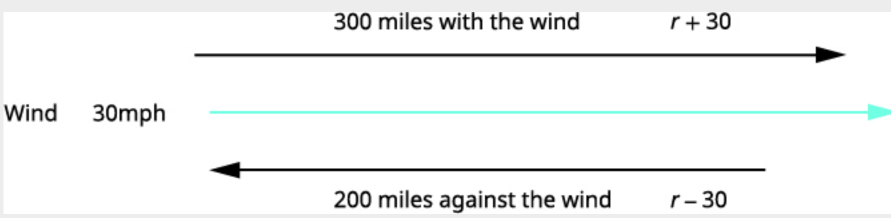
Example:
Exercise:

Problem:

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.	
We are looking for the speed of the airplane.	Let r = the speed of the airplane.
When the plane flies with the wind, the wind increases its speed and so the rate is $r + 30$.	
When the plane flies against the wind, the wind decreases its speed and the rate is $r - 30$.	

Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	<table><tr><th></th><th>Rate</th><th>·</th><th>Time</th><th>=</th><th>Distanc</th></tr><tr><td>Headwind</td><td>$r - 30$</td><td></td><td>$\frac{200}{r-30}$</td><td></td><td>200</td></tr><tr><td>Tailwind</td><td>$r + 30$</td><td></td><td>$\frac{300}{r+30}$</td><td></td><td>300</td></tr></table>
We know the times are equal and so we write our equation.	$\frac{200}{r-30} = \frac{300}{r+30}$
We multiply both sides by the LCD.	$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)($
Simplify.	$(r+30)(200) = (r-30)300$
	$200r + 6000 = 300r - 9000$
Solve.	$15000 = 100r$
Check. Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph, Tailwind $150 + 30 = 180$ mph $\frac{300}{180} = \frac{5}{3}$ hours Headwind $150 - 30 = 120$ mph $\frac{200}{120} = \frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

Note:
Exercise:

Problem:

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Solution:

Link's biking speed is 15 mph.

Note:
Exercise:

Problem:

Danica can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Danica's boat without a wind?

Solution:

The speed of Danica's boat is 17 mph.

In the next example, we will know the total time resulting from travelling different distances at different speeds.

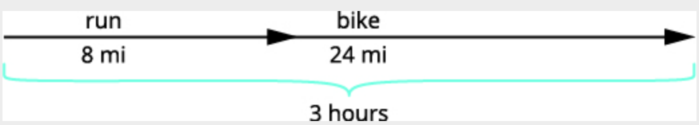
Example:
Exercise:

Problem:

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let r = Jazmine's running speed.
Her biking speed is 4 miles faster than her running speed.	$r + 4$ = her biking speed
The distances are given, enter them into the chart. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	(chart below)
Write a word sentence.	Her time plus the time biking is 3 hours.
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} \quad = \quad 3$
Solve.	

	Rate $\cdot$	Time =	Distance
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

	$$\begin{aligned} r(r+4)\left(\frac{8}{r}+\frac{24}{r+4}\right) &= 3\cdot r(r+4) \\ 8(r+4)+24r &= 3r(r+4) \\ 8r+32+24r &= 3r^2+12r \\ 32+32r &= 3r^2+12r \\ 0 &= 3r^2-20r-32 \\ 0 &= (3r+4)(r-8) \end{aligned}$$
	$(3r+4)=0 \quad (r-8)=0$
	$\cancel{r=-\frac{4}{3}} \quad r=8$
Check. A negative speed does not make sense in this problem, so $r=8$ is the solution. Is 8 mph a reasonable running speed? Yes. If Jazmine's running rate is 4, then her biking rate, $r+4$, which is $8+4=12$. Run 8 mph $\frac{8\text{ miles}}{8\text{ mph}} = 1$ hour Bike 12 mph $\frac{24\text{ miles}}{12\text{ mph}} = 2$ hours	
Total 3 hours.	Jazmine's running speed is 8 mph.

Note:
Exercise:

Problem:

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Solution:

Dennis's uphill speed was 10 mph and his downhill speed was 5 mph.

Note:
Exercise:

Problem:

Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Solution:

Joon's rate on the country roads is 50 mph.

Once again, we will use the uniform motion formula solved for the variable t.

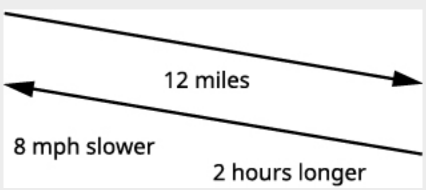
Example:
Exercise:

Problem:

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.	
We are looking for Hamilton's downhill speed.	Let h = Hamilton's downhill speed.
His uphill speed is 8 miles per hour slower. Enter the rates into the chart.	$h - 8$ = Hamilton's uphill speed
The distance is the same in both directions. 12 miles. Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	see chart below
Write a word sentence about the line.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation.	$\frac{12}{h-8} = \frac{12}{h} + 2$
Solve.	

	Rate	Time	Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

	$$\begin{aligned} h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\ 12h &= 12(h-8)+2h(h-8) \\ 12h &= 12h-96+2h^2-16h \\ 0 &= 2h^2-16h-96 \\ 0 &= 2(h^2-8h-48) \\ 0 &= 2(h-12)(h+4) \end{aligned}$$ $$\begin{aligned} h-12 &= 0 \quad & h+4 &= 0 \\ h &= 12 \quad & \cancel{h} &\cancel{= -4} \end{aligned}$$
Check. Is 12 mph a reasonable speed for biking downhill? Yes. $$\begin{aligned} &\text{Downhill} \quad 12\text{ mph} & \frac{12\text{ miles}}{12\text{ mph}} &= 1\text{ hour} \\ &\text{Uphill} \quad 12-8=4\text{ mph} & \frac{12\text{ miles}}{4\text{ mph}} &= 3\text{ hours.} \end{aligned}$$	
The uphill time is 2 hours more that the downhill time.	
	Hamilton's downhill speed is 12 mph.

Note:
Exercise:

Problem:

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

Solution:

Kayla's biking speed was 15 mph.

Note:
Exercise:

Problem:

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 20 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Solution:

Victoria jogged 6 mph on the flat trail.

Solve Work Applications

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

This is a typical 'work' application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete $\frac{1}{6}$ of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete $\frac{1}{12}$ of the job.

We will let t be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for t.

A chart will help us organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

<table>
<tr><td>Let t = the number of hours needed to complete the job together.</td><td></td></tr>
<tr><td>Enter the hours per job for Press #1, Press #2, and when they work together.

If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed.

Similarly find the part of the job completed/hours for Press #2 and when thet both together.

Write a word sentence.</td><td><table>
<tr><th></th><th>Number of hours to complete the job.</th><th>Part of job completed/hour</th></tr>
<tr><td>Press #1</td><td>6</td><td>$\frac{1}{6}$</td></tr>
<tr><td>Press #2</td><td>12</td><td>$\frac{1}{12}$</td></tr>
<tr><td>Together</td><td>t</td><td>$\frac{1}{t}$</td></tr>
</table></td></tr>
<tr><td></td><td>The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.</td></tr>
<tr><td>Translate into an equation.</td><td>Work completed by
$\underbrace{\text{Press \#1} + \text{Press \#2} = \text{Together}}$
$\frac{1}{6} \quad + \quad \frac{1}{12} \quad = \quad \frac{1}{t}$</td></tr>
<tr><td>Solve.</td><td>$\frac{1}{6}+\frac{1}{12}=\frac{1}{t}$</td></tr>
</table>

Mutiply by the LCD, $12t$	$12t\left(\frac{1}{6}+\frac{1}{12}\right)=12t\left(\frac{1}{t}\right)$
Simplify.	$2t+t=12$ $3t=12$ $t=4$
	When both presses are running it takes 4 hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it alone.

Example:
Exercise:

Problem:

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

Solution:

This is a 'work' application. A chart will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

Let t be the number of hours needed to paint the room together.	
Enter the hours per job for Pete, Alicia, and when they work together. In 1 hour working together, they have completed $\frac{1}{t}$ of the job. Similarly, find the part of the job completed/hour by Pete and then by Alicia.	(see table below)
Write a word sentence.	The work completed by Pete plus the work completed by Alicia equals the total work completed. Work completed by:

	Number of complete hours to the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

	$\underbrace{\text{Pete} + \text{Alicia} = \text{Together}}$ $\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$ $\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$
Multiply by the LCD, 40*t*.	$40t\left(\frac{1}{10} + \frac{1}{8}\right) = 40t\left(\frac{1}{t}\right)$
Distribute.	$40t \cdot \frac{1}{10} + 40t \cdot \frac{1}{8} = 40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t + 5t = 40$ $9t = 40$ $t = \frac{40}{9}$
We’ll write as a mixed number so that we can convert it to hours and minutes.	$t = 4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t = 4$ hours $+ \frac{4}{9}$ (60 minutes)
Multiply, and then round to the nearest minute.	$t = 4$ hours $+ 27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Note:
Exercise:

Problem:

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Solution:

When the two gardeners work together it takes 2 hours and 24 minutes.

Note:
Exercise:

Problem:

Daria can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Solution:

When Daria and her mother work together it takes 2 hours and 6 minutes.

Example:
Exercise:

Problem:

Ra’shon can clean the house in 7 hours. When his sister helps him it takes 3 hours. How long does it take his sister when she cleans the house alone?

Solution:

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take Ra’shon’s sister to complete the job by herself.

Let s be the number of hours Ra’shon’s sister takes to clean the house alone.	
Enter the hours per job for Ra’shon, his sister, and when they work together. If Ra’shon takes 7 hours, then in 1 hour $\frac{1}{7}$ of the job is completed. If Ra’shon’s sister takes s hours, then in 1 hour $\frac{1}{s}$ of the job is completed.	<table><tr><th></th><th>Number of hours to clean the house</th><th>Part of job completed/hour</th></tr><tr><td>Ra’shon</td><td>7</td><td>$\frac{1}{7}$</td></tr><tr><td>His sister</td><td>s</td><td>$\frac{1}{s}$</td></tr><tr><td>Together</td><td>3</td><td>$\frac{1}{3}$</td></tr></table>
Write a word sentence.	The part completed by Ra’shon plus the part by his sister equals the amount completed together.
Translate to an equation.	Work completed by Ra’shon + His sister = Together $\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$
Solve.	$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$
Multiply by the LCD, $21s$.	

	$$21s\left(\frac{1}{7}+\frac{1}{s}\right)=\left(\frac{1}{3}\right)21s$$ $$3s+21=7s$$
Simplify.	$$-4s=-21$$ $$s=\frac{-21}{-4}=\frac{21}{4}$$
Write as a mixed number to convert it to hours and minutes.	$$s=5\frac{1}{4}\text{ hours}$$
There are 60 minutes in 1 hour.	$$s=5\text{ hours}+\frac{1}{4}(60\text{ minutes})$$ $$s=5\text{ hours}+15\text{ minutes}$$
	It would take Ra'shon's sister 5 hours and 15 minutes to clean the house alone.

Note:
Exercise:

Problem:

Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

Solution:

Kristina can paint the room in 12 hours.

Note:
Exercise:

Problem:

Tracy can lay a slab of concrete in 3 hours, with Jordan's help they can do it in 2 hours. If Jordan works alone, how long will it take?

Solution:

It will take Jordan 6 hours.

Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Lindsay gets paid \$15 per hour at her job. If we let s be her salary and h be the number of hours she has worked, we could model this situation with the equation
Equation:

$$s = 15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Note:
Direct Variation
For any two variables x and y, y varies directly with x if
Equation:

$$y = kx, \text{ where } k \neq 0$$

The constant k is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values of y for other values of x.

We'll list the steps here.

Note:
Solve direct variation problems.

Write the formula for direct variation.
Substitute the given values for the variables.
Solve for the constant of variation.
Write the equation that relatesxandyusing the constant of variation.

Now we'll solve an application of direct variation.

Example:
Exercise:

Problem:

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates c and m. ⓑ How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution:

ⓐ

	The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.
Write the formula for direct variation.	$y = kx$
We will use c in place of y and m in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$ $17.5 = k$
Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

ⓑ

	Find c when $m = 25$.
Write the equation that relates c and m.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$

Simplify.	$c = 437.5$
	Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

Note:
Exercise:

Problem:

The number of calories, *c*, burned varies directly with the amount of time, *t*, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates *c* and *t*. ⓑ How many calories would he burn if he exercises for 90 minutes?

Solution:

ⓐ $c = 4.8t$ ⓑ He would burn 432 calories.

Note:
Exercise:

Problem:

The distance a moving body travels, *d*, varies directly with time, *t*, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates *d* and *t*. ⓑ How many miles would it travel in 5 hours?

Solution:

ⓐ $d = 50t$ ⓑ It would travel 250 miles.

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Note:
Inverse Variation
For any two variables *x* and *y*, *y* varies inversely with *x* if
Equation:

$$y=\frac{k}{x}, \text{ where } k\neq 0$$

The constant *k* is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of *x* is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

Note:
Solve inverse variation problems.

Write the formula for inverse variation.
Substitute the given values for the variables.
Solve for the constant of variation.
Write the equation that relates*x*and*y*using the constant of variation.

Example:
Exercise:

Problem:

The frequency of a guitar string varies inversely with its length. A 26 in.-long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation. ⓑ How many vibrations per second will there be if the string’s length is reduced to 20 inches by putting a finger on a fret?

Solution:

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let *f* = frequency. *L* = length
Write the formula for inverse variation.	$y=\frac{k}{x}$
We will use *f* in place of *y* and *L* in place of *x*.	$f=\frac{k}{L}$

Substitute the given values for the variables.	$f = 440$ when $L = 26$ $440 = \frac{k}{26}$
Solve for the constant of variation	$26(440) = 26\left(\frac{k}{26}\right)$ $11{,}440 = k$
Write the equation that relates *f* and *L*.	$f = \frac{k}{L}$
Substitute the constant of variation	$f = \frac{11{,}440}{L}$

ⓑ

	Find f when $L = 20$.
Write the equation that relates $f$and L.	$f = \frac{11{,}440}{L}$
Substitute the given value for L.	$f = \frac{11{,}440}{20}$
Simplify.	$f = 572$
	A 20″-guitar string has frequency 572 vibrations per second.

Note:
Exercise:

Problem:

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

ⓐ Write the equation of variation. ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

Solution:

ⓐ $h = \frac{130}{t}$ ⓑ $1\frac{2}{3}$ hours

Note:
Exercise:

Problem:

Xander's new business found that the daily demand for its product was inversely proportional to the price, p. When the price is \$5, the demand is 700 units.

ⓐ Write the equation of variation. ⓑ What is the demand if the price is raised to \$7?

Solution:

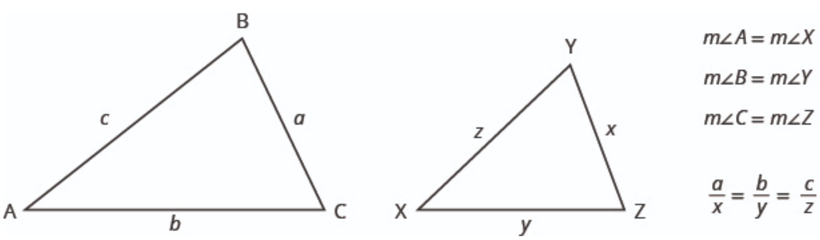
ⓐ $x = \frac{3500}{p}$ ⓑ 500 units

Note:
Access this online resource for additional instruction and practice with applications of rational expressions

- Applications of Rational Expressions

Key Concepts

- A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$. The proportion is read "*a* is to *b* as *c* is to *d*."
- **Property of Similar Triangles**
  If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



- **Direct Variation**
  - For any two variables *x* and *y*, *y* varies directly with *x* if $y = kx$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve direct variation problems.

    Write the formula for direct variation.
    Substitute the given values for the variables.
    Solve for the constant of variation.
    Write the equation that relates x and y.

- **Inverse Variation**
  - For any two variables *x* and *y*, *y* varies inversely with *x* if $y = \frac{k}{x}$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve inverse variation problems.

    Write the formula for inverse variation.
    Substitute the given values for the variables.

Solve for the constant of variation.
Write the equation that relates x and y.

Practice Makes Perfect

Solve Proportions

In the following exercises, solve each proportion.

Exercise:

Problem: $\frac{x}{56} = \frac{7}{8}$

Solution:

$x = 49$

Exercise:

Problem: $\frac{56}{72} = \frac{y}{9}$

Exercise:

Problem: $\frac{98}{154} = \frac{-7}{p}$

Solution:

$p = -11$

Exercise:

Problem: $\frac{72}{156} = \frac{-6}{q}$

Exercise:

Problem: $\frac{a}{a+12} = \frac{4}{7}$

Solution:

$a = 16$

Exercise:

Problem: $\frac{b}{b-16} = \frac{11}{9}$

Exercise:

Problem: $\frac{m+90}{25} = \frac{m+30}{15}$

Solution:

$m = 60$

Exercise:

Problem: $\frac{n+10}{4} = \frac{40-n}{6}$

Exercise:

Problem: $\frac{2p+4}{8}=\frac{p+18}{6}$

Solution:

$p=30$

Exercise:

Problem: $\frac{q-2}{2}=\frac{2q-7}{18}$

In the following exercises, solve.

Exercise:

Problem:

Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?
ⓑ Has Kevin met his target heart rate?

Solution:

ⓐ 162 beats per minute
ⓑ yes

Exercise:

Problem: Jesse's car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

Exercise:

Problem:

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

Solution:

9 ml

Exercise:

Problem:

A veterinarian prescribed Sunny, a 65-pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

Exercise:

Problem:

A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

Solution:

159 calories

Exercise:

Problem:

One 12-ounce can of soda has 150 calories. If Josiah drinks the big 32-ounce size from the local mini-mart, how many calories does he get?

Exercise:

Problem:

Kyra is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.3 Canadian. How many Canadian dollars will she get for her trip?

Solution:

325 Canadian dollars

Exercise:

Problem:

Maurice is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

Exercise:

Problem:

Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

Solution:

3 cups

Exercise:

Problem:

Sonya drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

Exercise:

Problem:

Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

Solution:

4 bags

Exercise:

Problem:

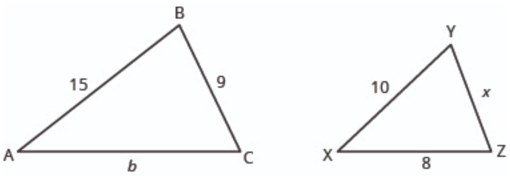
An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, the triangles are similar. Find the length of the indicated side.

Exercise:

Problem:



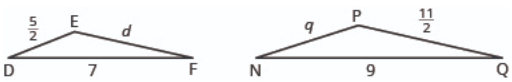
ⓐ side x
ⓑ side b

Solution:

ⓐ 6 ⓑ 12

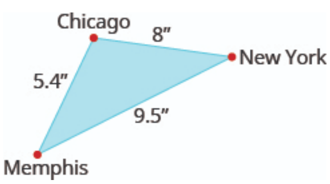
Exercise:

Problem:



ⓐ side d
ⓑ side q

In the following exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle. The actual distance from New York to Chicago is 800 miles.



Exercise:

Problem: Find the actual distance from New York to Memphis.

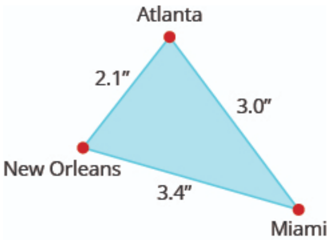
Solution:

950 miles

Exercise:

Problem: Find the actual distance from Chicago to Memphis.

In the following exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle. The actual distance from Atlanta to New Orleans is 420 miles.



Exercise:

Problem: Find the actual distance from New Orleans to Miami.

Solution:

680 miles

Exercise:

Problem: Find the actual distance from Atlanta to Miami.

In the following exercises, answer each question.

Exercise:

Problem:

A 2-foot-tall dog casts a 3-foot shadow at the same time a cat casts a one foot shadow. How tall is the cat ?

Solution:

$\frac{2}{3}$ foot (8 in.)

Exercise:

Problem:

Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

Exercise:

Problem:

The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven-foot shadow. How long is the windmill's shadow?

Solution:

247.3 feet

Exercise:

Problem:

The height of the Statue of Liberty is 305 feet. Nikia, who is standing next to the statue, casts a 6-foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Solve Uniform Motion Applications

In the following exercises, solve the application problem provided.

Exercise:

Problem:

Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35-mph headwind in the same amount of time it can travel 702 miles with a 35-mph tailwind. Find the speed of the helicopter.

Solution:

160 mph

Exercise:

Problem:

A private jet can fly 1,210 miles against a 25-mph headwind in the same amount of time it can fly 1694 miles with a 25-mph tailwind. Find the speed of the jet.

Exercise:

Problem:

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Solution:

29 mph

Exercise:

Problem:

Darrin can skateboard 2 miles against a 4-mph wind in the same amount of time he skateboards 6 miles with a 4-mph wind. Find the speed Darrin skateboards with no wind.

Exercise:

Problem:

Jane spent 2 hours exploring a mountain with a dirt bike. First, she rode 40 miles uphill. After she reached the peak she rode for 12 miles along the summit. While going uphill, she went 5 mph slower than when she was on the summit. What was her rate along the summit?

Solution:

30 mph

Exercise:

Problem:

Laney wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Exercise:

Problem:

Byron wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Solution:

20 mph

Exercise:

Problem:

Nancy took a 3-hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

Exercise:

Problem:

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, what was his uphill rate?

Solution:

4 mph

Exercise:

Problem:

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Exercise:

Problem:

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Solution:

60 mph

Exercise:

Problem:

Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel.

Exercise:

Problem:

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Solution:

650 mph

Exercise:

Problem:

Jim’s speedboat can travel 20 miles upstream against a 3-mph current in the same amount of time it travels 22 miles downstream with a 3-mph current speed . Find the speed of the Jim’s boat.

Exercise:

Problem:

Hazel needs to get to her granddaughter’s house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Solution:

50 mph

Exercise:

Problem:

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

Exercise:

Problem:

When driving the 9-hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Solution:

50 mph

Exercise:

Problem:

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Exercise:

Problem:

Dana enjoys taking her dog for a walk, but sometimes her dog gets away, and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Solution:

4.2 mph

Exercise:

Problem:

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe’s speed?

Solve Work Applications

Exercise:

Problem:

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Solution:

2 hours

Exercise:

Problem:

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Exercise:

Problem:

Mia can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Solution:

2 hours and 44 minutes

Exercise:

Problem:

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Exercise:

Problem:

Josephine can correct her students test papers in 5 hours, but if her teacher’s assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Solution:

7 hours and 30 minutes

Exercise:

Problem:

Washing his dad’s car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take Levi’s dad to wash the car by himself?

Exercise:

Problem:

At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Solution:

10 min

Exercise:

Problem:

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solve Direct Variation Problems

In the following exercises, solve.

Exercise:

Problem: If y varies directly as x and $y = 14,$ when $x = 3$. find the equation that relates x and y.

Solution:

$y = \frac{14}{3}x$

Exercise:

Problem: If a varies directly as b and $a = 16,$ when $b = 4$. find the equation that relates a and b.

Exercise:

Problem: If p varies directly as q and $p = 9.6,$ when $q = 3$. find the equation that relates p and q.

Solution:

$p = 3.2q$

Exercise:

Problem: If v varies directly as w and $v = 8,$ when $w = \frac{1}{2}$. find the equation that relates v and w.

Exercise:

Problem:

The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him $50 to buy 20 gallons of gas.

ⓐ Write the equation that relates P and g.
ⓑ How much would 33 gallons cost Eric?

Solution:

ⓐ $P = 2.5g$ ⓑ $\$82.50$

Exercise:

Problem:

Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates d and v.
ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

Exercise:

Problem:

The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume.
ⓑ What is the volume of this liquid if its mass is 128 kilograms?

Solution:

ⓐ $m = 8v$ ⓑ 16 liters

Exercise:

Problem:

The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight.
ⓑ What weight of watermelon would stretch the spring 6 inches?

Exercise:

Problem:

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section.
ⓑ What load will a beam with a 10-inch diagonal support?

Solution:

ⓐ $L = 3d^2$ ⓑ300 pounds

Exercise:

Problem:

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.
ⓑ What is the area of a personal pizza with a radius 4 inches?

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise:

Problem: If y varies inversely with x and $y = 5$ when $x = 4$, find the equation that relates x and y.

Solution:

$y = \frac{20}{x}$

Exercise:

Problem: If p varies inversely with q and $p = 2$ when $q = 1$, find the equation that relates p and q.

Exercise:

Problem: If v varies inversely with w and $v = 6$ when $w = \frac{1}{2}$, find the equation that relates v and w.

Solution:

$v = \frac{3}{w}$

Exercise:

Problem: If a varies inversely with b and $a = 12$ when $b = \frac{1}{3}$, find the equation that relates a and b.

In the following exercises, write an inverse variation equation to solve the following problems.

Exercise:

Problem:

The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds getting 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car's weight.
ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

Solution:

ⓐ $g = \frac{92,400}{w}$ ⓑ 16.8 mpg

Exercise:

Problem: A car's value varies inversely with its age. Jackie bought a 10-year-old car for $2,400.

ⓐ Write the equation that relates the car's value to its age.
ⓑ What will be the value of Jackie's car when it is 15 years old?

Exercise:

Problem:

The time required to empty a tank varies inversely as the rate of pumping. It took Ada 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute).

ⓐ Write the equation that relates the number of hours to the pump rate.
ⓑ How long would it take Ada to pump her basement if she used a pump rated at 400 gpm?

Solution:

ⓐ $t = \frac{1000}{r}$ ⓑ 2.5 hours

Exercise:

Problem:

On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency. ⓑ What is the frequency of a 10 inch string?

Exercise:

Problem:

Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had four cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing.
ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

Solution:

ⓐ $c = \frac{2}{t}$ ⓑ 1 cavity

Exercise:

Problem:

Boyle's law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume.
ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

Exercise:

Problem:

The cost of a ride service varies directly with the distance traveled. It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, c, with the number of miles, m.
ⓑ What would it cost to travel 22 miles with this service?

Solution:

ⓐ $c = 2.5m$ ⓑ $55

Exercise:

Problem:

The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, h, with the speed, s.
ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

Exercise:

Problem:

Marisol solves the proportion $\frac{144}{a} = \frac{9}{4}$ by 'cross multiplying,' so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in [link].

Solution:

Answers will vary.

Exercise:

Problem:

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3} + \frac{1}{4} = \frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Exercise:

Problem: In your own words, explain the difference between direct variation and inverse variation.

Solution:

Answers will vary.

Exercise:

Problem: Make up an example from your life experience of inverse variation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			
solve uniform motion applications.			
solve work applications.			
solve direct variation problems.			
solve inverse variation problems.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

proportion
: When two rational expressions are equal, the equation relating them is called a proportion.

similar figures
: Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

Solve Rational Inequalities
By the end of this section, you will be able to:

- Solve rational inequalities
- Solve an inequality with rational functions

Note:
Before you get started, take this readiness quiz.

1. Find the value of $x-5$ when ⓐ $x=6$ ⓑ $x=-3$ ⓒ $x=5$.
   If you missed this problem, review [link].
2. Solve: $8-2x<12$.
   If you missed this problem, review [link].
3. Write in interval notation: $-3\le x<5$.
   If you missed this problem, review [link].

Solve Rational Inequalities

We learned to solve linear inequalities after learning to solve linear equations. The techniques were very much the same with one major exception. When we multiplied or divided by a negative number, the inequality sign reversed.

Having just learned to solve rational equations we are now ready to solve rational inequalities. A **rational inequality** is an inequality that contains a rational expression.

Note:
Rational Inequality
A **rational inequality** is an inequality that contains a rational expression.

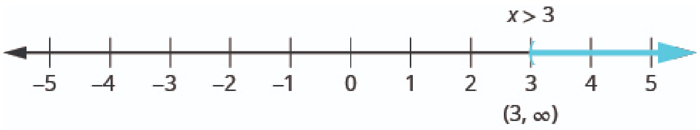
Inequalities such as $\frac{3}{2x}>1,\ \ \frac{2x}{x-3}<4,\ \ \frac{2x-3}{x-6}\ge x$, and $\frac{1}{4}-\frac{2}{x^2}\le\frac{3}{x}$ are rational inequalities as they each contain a rational expression.

When we solve a rational inequality, we will use many of the techniques we used solving linear inequalities. We especially must remember that when we multiply or divide by a negative number, the inequality sign must reverse.

Another difference is that we must carefully consider what value might make the rational expression undefined and so must be excluded.

When we solve an equation and the result is $x=3$, we know there is one solution, which is 3.

When we solve an inequality and the result is $x>3$, we know there are many solutions. We graph the result to better help show all the solutions, and we start with 3. Three becomes a **critical point** and then we decide whether to shade to the left or right of it. The numbers to the right of 3 are larger than 3, so we shade to the right.



To solve a rational inequality, we first must write the inequality with only one quotient on the left and 0 on the right.

Next we determine the critical points to use to divide the number line into intervals. A **critical point** is a number which make the rational expression zero or undefined.

We then will evaluate the factors of the numerator and denominator, and find the quotient in each interval. This will identify the interval, or intervals, that contains all the solutions of the rational inequality.

We write the solution in interval notation being careful to determine whether the endpoints are included.

Example:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{x-1}{x+3} \geq 0$.

Solution:

Step 1. Write the inequality as one quotient on the left and zero on the right.

Our inequality is in this form. $\frac{x-1}{x+3} \geq 0$

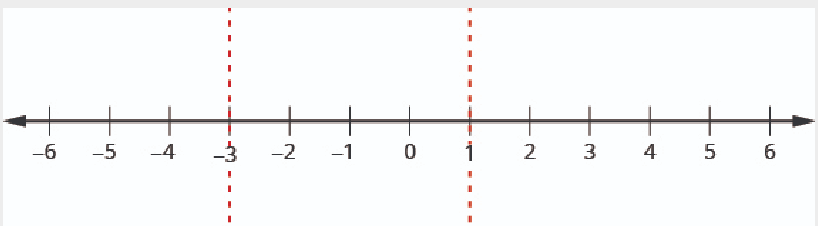
Step 2. Determine the critical points—the points where the rational expression will be zero or undefined.

The rational expression will be zero when the numerator is zero. Since $x-1=0$ when $x=1$, then 1 is a critical point.

The rational expression will be undefined when the denominator is zero. Since $x+3=0$ when $x=-3$, then -3 is a critical point.

The critical points are 1 and -3.

Step 3. Use the critical points to divide the number line into intervals.



The number line is divided into three intervals:

$$(-\infty,-3) \qquad (-3,1) \qquad (1,\infty)$$

Step 4. Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.

To find the sign of each factor in an interval, we choose any point in that interval and use it as a test point. Any point in the interval will give the expression the same sign, so we can choose any point in the interval.

Equation:

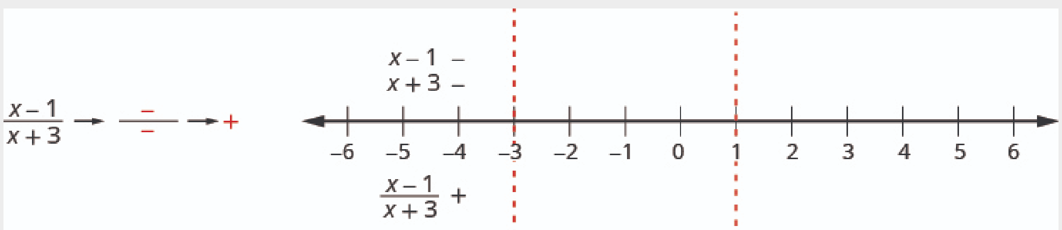
$$\textbf{Interval } (-\infty, -3)$$

The number -4 is in the interval $(-\infty, -3)$. Test $x = -4$ in the expression in the numerator and the denominator.

the numerator	$x-1$	the denominator	$x+3$
	$-4-1$		$-4+3$
	-5		3
	Negative		Negative

Above the number line, mark the factor $x - 1$ negative and mark the factor $x + 3$ negative.

Since a negative divided by a negative is positive, mark the quotient positive in the interval $(-\infty, -3)$.



Equation:

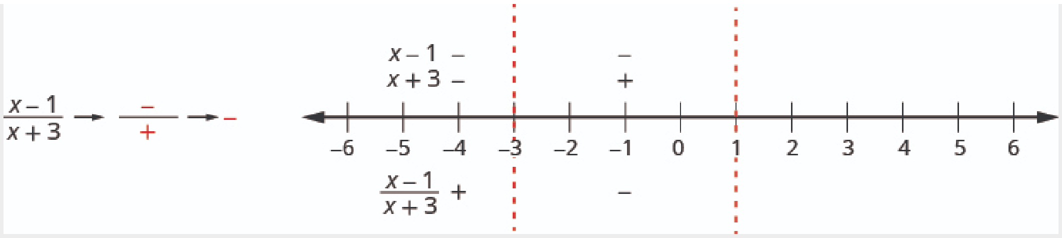
$$\textbf{Interval } (-3, 1)$$

The number 0 is in the interval $(-3, 1)$. Test $x = 0$.

the numerator	$x-1$	the denominator	$x+3$
	$0-1$		$0+3$
	-1		3
	Negative		Positive

Above the number line, mark the factor $x - 1$ negative and mark $x + 3$ positive.

Since a negative divided by a positive is negative, the quotient is marked negative in the interval $(-3, 1)$.



Equation:

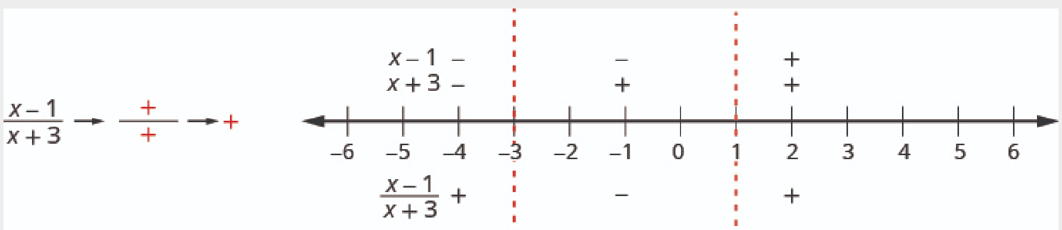
$$\textbf{Interval } (1, \infty)$$

The number 2 is in the interval $(1, \infty)$. Test $x = 2$.

the numerator	$x-1$	the denominator	$x+3$
	$2-1$		$2+3$
	1		5
	Positive		Positive

Above the number line, mark the factor $x - 1$ positive and mark $x + 3$ positive.

Since a positive divided by a positive is positive, mark the quotient positive in the interval $(1, \infty)$.



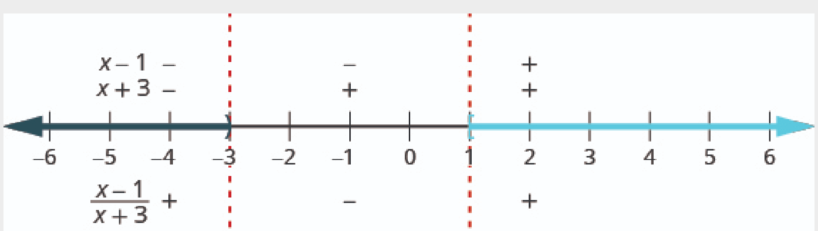
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty, -3)$ and $(1, \infty)$ are solutions.

But what about the critical points?

The critical point $x = -3$ makes the denominator 0, so it must be excluded from the solution and we mark it with a parenthesis.

The critical point $x = 1$ makes the whole rational expression 0. The inequality requires that the rational expression be greater than or equal to. So, 1 is part of the solution and we will mark it with a bracket.



Recall that when we have a solution made up of more than one interval we use the union symbol, $\cup$, to connect the two intervals. The solution in interval notation is $(-\infty, -3) \cup [1, \infty)$.

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{x-2}{x+4} \geq 0$.

Solution:

$(-\infty, -4) \cup [2, \infty)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{x+2}{x-4} \geq 0$.

Solution:

$(-\infty, -2] \cup (4, \infty)$

We summarize the steps for easy reference.

Note:
Solve a rational inequality.

Write the inequality as one quotient on the left and zero on the right.
Determine the critical points–the points where the rational expression will be zero or undefined.
Use the critical points to divide the number line into intervals.
Test a value in each interval. Above the number line show the sign of each factor of the numerator and denominator in each interval. Below the number line show the sign of the quotient.
Determine the intervals where the inequality is correct. Write the solution in interval notation.

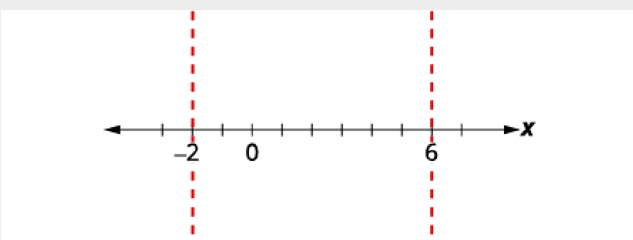
The next example requires that we first get the rational inequality into the correct form.

Example:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{4x}{x-6} < 1$.

Solution:

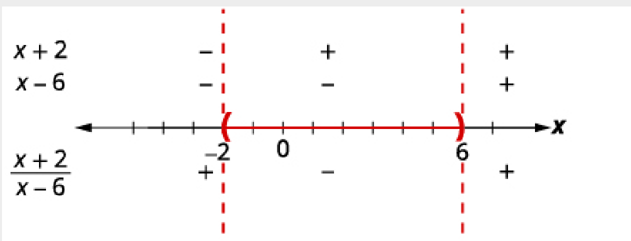
	$\frac{4x}{x-6} < 1$	
Subtract 1 to get zero on the right.	$\frac{4x}{x-6} - 1 < 0$	
Rewrite 1 as a fraction using the LCD.	$\frac{4x}{x-6} - \frac{x-6}{x-6} < 0$	
Subtract the numerators and place the difference over the common denominator.	$\frac{4x-(x-6)}{x-6} < 0$	
Simplify.	$\frac{3x+6}{x-6} < 0$	
Factor the numerator to show all factors.	$\frac{3(x+2)}{x-6} < 0$	
Find the critical points.		
The quotient will be zero when the numerator is zero. The quotient is undefined when the denominator is zero.	$\begin{aligned} x+2 &= 0 \\ x &= -2 \end{aligned}$ $\quad$ $\begin{aligned} x-6 &= 0 \\ x &= 6 \end{aligned}$	
Use the critical points to divide the number line into intervals.		



Test a value in each interval.

	$(-\infty, -2)$	$(-2, 6)$	$(6, \infty)$
$x+2$	$x+2$ $-3+2$ -1 $-$	$x+2$ $0+2$ 2 $+$	$x+2$ $7+2$ 9 $+$
$x-6$	$x-6$ $-3-6$ -9 $-$	$x-6$ $0-6$ -6 $-$	$x-6$ $7-6$ 1 $+$

Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.



Determine the intervals where the inequality is correct. We want the quotient to be negative, so the solution includes the points between −2 and 6. Since the inequality is strictly less than, the endpoints are not included.

We write the solution in interval notation as $(-2, 6)$.

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{3x}{x-3} < 1$.

Solution:

$\left(-\frac{3}{2}, 3\right)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{3x}{x-4} < 2$.

Solution:

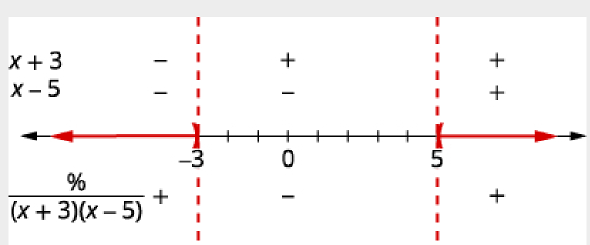
$(-8, 4)$

In the next example, the numerator is always positive, so the sign of the rational expression depends on the sign of the denominator.

Example:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{5}{x^2-2x-15} > 0$.

Solution:

The inequality is in the correct form.	$\frac{5}{x^2-2x-15} > 0$
Factor the denominator.	$\frac{5}{(x+3)(x-5)} > 0$
Find the critical points. The quotient is 0 when the numerator is 0. Since the numerator is always 5, the quotient cannot be 0.	
The quotient will be undefined when the denominator is zero.	$(x+3)(x-5) = 0$ $x = -3,\ x = 5$
Use the critical points to divide the number line into intervals.	
	
Test values in each interval. Above the number line show the sign of each	

factor of the denominator in each interval. Below the number line, show the sign of the quotient.	
Write the solution in interval notation.	$(-\infty, -3) \cup (5, \infty)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{1}{x^2+2x-8} > 0$.

Solution:

$(-\infty, -4) \cup (2, \infty)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{3}{x^2+x-12} > 0$.

Solution:

$(-\infty, -4) \cup (3, \infty)$

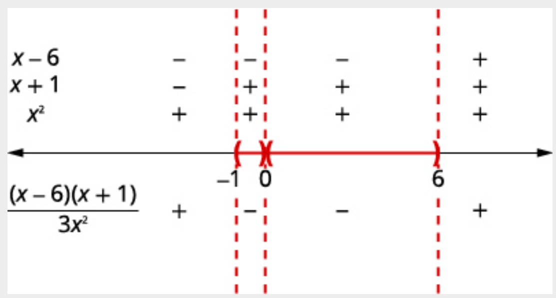
The next example requires some work to get it into the needed form.

Example:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{1}{3} - \frac{2}{x^2} < \frac{5}{3x}$.

Solution:

	$\frac{1}{3} - \frac{2}{x^2} < \frac{5}{3x}$	
Subtract $\frac{5}{3x}$ to get zero on the right.	$\frac{1}{3} - \frac{2}{x^2} - \frac{5}{3x} < 0$	

Rewrite to get each fraction with the LCD $3x^2$.	$\frac{1 \cdot x^2}{3 \cdot x^2} - \frac{2 \cdot 3}{x^2 \cdot 3} - \frac{5 \cdot x}{3x \cdot x} < 0$
Simplify.	$\frac{x^2}{3x^2} - \frac{6}{3x^2} - \frac{5x}{3x^2} < 0$
Subtract the numerators and place the difference over the common denominator.	$\frac{x^2-5x-6}{3x^2} < 0$
Factor the numerator.	$\frac{(x-6)(x+1)}{3x^2} < 0$
Find the critical points.	$\begin{aligned} 3x^2 &= 0 \\ x &= 0 \end{aligned} \quad \begin{aligned} x-6 &= 0 \\ x &= 6 \end{aligned} \quad \begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$
Use the critical points to divide the number line into intervals.	
	
Above the number line show the sign of each factor in each interval. Below the number line, show the sign of the quotient.	
Since, 0 is excluded, the solution is the two intervals, $(-1, 0)$ and $(0, 6)$.	$(-1, 0) \cup (0, 6)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{1}{2} + \frac{4}{x^2} < \frac{3}{x}$.

Solution:

$(2, 4)$

Note:
Exercise:

Problem: Solve and write the solution in interval notation: $\frac{1}{3}+\frac{6}{x^2}<\frac{3}{x}$.

Solution:

$(3,6)$

Solve an Inequality with Rational Functions

When working with rational functions, it is sometimes useful to know when the function is greater than or less than a particular value. This leads to a rational inequality.

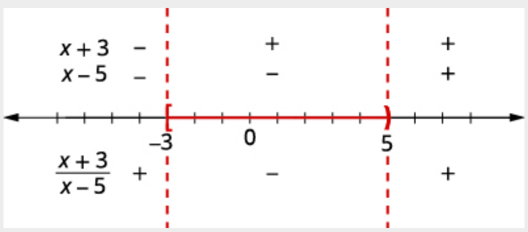
Example:
Exercise:

Problem:

Given the function $R(x)=\frac{x+3}{x-5}$, find the values of *x* that make the function less than or equal to 0.

Solution:

We want the function to be less than or equal to 0.

	$R(x)\le 0$
Substitute the rational expression for $R(x)$.	$\frac{x+3}{x-5}\le 0 \qquad x\ne 5$
Find the critical points.	$\begin{aligned} x+3 &= 0 \\ x &= -3 \end{aligned} \qquad \begin{aligned} x-5 &= 0 \\ x &= 5 \end{aligned}$
Use the critical points to divide the number line into intervals.	
	

Test values in each interval. Above the number line, show the sign of each factor in each interval. Below the number line, show the sign of the quotient	
Write the solution in interval notation. Since 5 is excluded we, do not include it in the interval.	$[-3, 5)$

Note:
Exercise:

Problem:

Given the function $R(x) = \frac{x-2}{x+4}$, find the values of *x* that make the function less than or equal to 0.

Solution:

$(-4, 2]$

Note:
Exercise:

Problem:

Given the function $R(x) = \frac{x+1}{x-4}$, find the values of *x* that make the function less than or equal to 0.

Solution:

$[-1, 4)$

In economics, the function $C(x)$ is used to represent the cost of producing *x* units of a commodity. The average cost per unit can be found by dividing $C(x)$ by the number of items x. Then, the average cost per unit is $c(x) = \frac{C(x)}{x}$.

Example:
Exercise:

Problem:

The function $C(x) = 10x + 3000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $40.

Solution:

ⓐ

$$C(x) = 10x + 3000$$

The average cost function is $c(x) = \frac{C(x)}{x}$.

To find the average cost function, divide the cost function by x.

$$c(x) = \frac{C(x)}{x}$$

$$c(x) = \frac{10x+3000}{x}$$

The average cost function is $c(x) = \frac{10x+3000}{x}$.

ⓑ

We want the function $c(x)$ to be less than 40.

$$c(x) < 40$$

Substitute the rational expression for $c(x)$.

$$\frac{10x+3000}{x} < 40 \quad x \neq 0$$

Subtract 40 to get 0 on the right.

$$\frac{10x+3000}{x} - 40 < 0$$

Rewrite the left side as one quotient by finding the LCD and performing the subtraction.

$$\frac{10x+3000}{x} - 40\left(\frac{x}{x}\right) < 0$$

$$\frac{10x+3000}{x} - \frac{40x}{x} < 0$$

$$\frac{10x+3000-40x}{x} < 0$$

$$\frac{-30x+3000}{x} < 0$$

Factor the numerator to show all factors.

$$\frac{-30(x-100)}{x} < 0$$

$$-30(x-100) = 0 \qquad x = 0$$

Find the critical points.

$$-30 \neq 0 \quad x - 100 = 0$$

$$x = 100$$

More than 100 items must be produced to keep the average cost below \$40 per item.

Note:
Exercise:

Problem:

The function $C(x) = 20x + 6000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than \$60?

Solution:

ⓐ $c(x) = \frac{20x+6000}{x}$
ⓑ More than 150 items must be produced to keep the average cost below \$60 per item.

Note:

Exercise:

Problem:

The function $C(x)=5x+900$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than $20?

Solution:

ⓐ $c(x)=\frac{5x+900}{x}$ ⓑ More than 60 items must be produced to keep the average cost below $20 per item.

Key Concepts

- **Solve a rational inequality.**

  Write the inequality as one quotient on the left and zero on the right.
  Determine the critical points–the points where the rational expression will be zero or undefined.
  Use the critical points to divide the number line into intervals.
  Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.
  Determine the intervals where the inequality is correct. Write the solution in interval notation.

Section Exercises

Practice Makes Perfect

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

Exercise:

Problem: $\frac{x-3}{x+4}\geq 0$

Solution:

$(-\infty,-4)\cup[3,\infty)$

Exercise:

Problem: $\frac{x+6}{x-5}\geq 0$

Exercise:

Problem: $\frac{x+1}{x-3}\leq 0$

Solution:

$[-1,3)$

Exercise:

Problem: $\frac{x-4}{x+2} \leq 0$

Exercise:

Problem: $\frac{x-7}{x-1} > 0$

Solution:

$(-\infty, 1) \cup (7, \infty)$

Exercise:

Problem: $\frac{x+8}{x+3} > 0$

Exercise:

Problem: $\frac{x-6}{x+5} < 0$

Solution:

$(-5, 6)$

Exercise:

Problem: $\frac{x+5}{x-2} < 0$

Exercise:

Problem: $\frac{3x}{x-5} < 1$

Solution:

$\left(-\frac{5}{2}, 5\right)$

Exercise:

Problem: $\frac{5x}{x-2} < 1$

Exercise:

Problem: $\frac{6x}{x-6} > 2$

Solution:

$(-\infty, -3) \cup (6, \infty)$

Exercise:

Problem: $\frac{3x}{x-4} > 2$

Exercise:

Problem: $\frac{2x+3}{x-6} \leq 1$

Solution:

$[-9, 6)$

Exercise:

Problem: $\frac{4x-1}{x-4} \leq 1$

Exercise:

Problem: $\frac{3x-2}{x-4} \geq 2$

Solution:

$(-\infty, -6] \cup (4, \infty)$

Exercise:

Problem: $\frac{4x-3}{x-3} \geq 2$

Exercise:

Problem: $\frac{1}{x^2+7x+12} > 0$

Solution:

$(-\infty, -3) \cup (-4, \infty)$

Exercise:

Problem: $\frac{1}{x^2-4x-12} > 0$

Exercise:

Problem: $\frac{3}{x^2-5x+4} < 0$

Solution:

$(1, 4)$

Exercise:

Problem: $\frac{4}{x^2+7x+12} < 0$

Exercise:

Problem: $\frac{2}{2x^2+x-15} \geq 0$

Solution:

$(-\infty, -3) \cup (\frac{5}{2}, \infty)$

Exercise:

Problem: $\frac{6}{3x^2-2x-5} \geq 0$

Exercise:

Problem: $\frac{-2}{6x^2-13x+6} \leq 0$

Solution:

$(-\infty, \frac{2}{3}) \cup (\frac{3}{2}, \infty)$

Exercise:

Problem: $\frac{-1}{10x^2+11x-6} \leq 0$

Exercise:

Problem: $\frac{1}{2} + \frac{12}{x^2} > \frac{5}{x}$

Solution:

$(-\infty, 0) \cup (0, 4) \cup (6, \infty)$

Exercise:

Problem: $\frac{1}{3} + \frac{1}{x^2} > \frac{4}{3x}$

Exercise:

Problem: $\frac{1}{2} - \frac{4}{x^2} \leq \frac{1}{x}$

Solution:

$[-2, 0) \cup (0, 4]$

Exercise:

Problem: $\frac{1}{2} - \frac{3}{2x^2} \geq \frac{1}{x}$

Exercise:

Problem: $\frac{1}{x^2-16} < 0$

Solution:

$(-4, 4)$

Exercise:

Problem: $\frac{4}{x^2-25} > 0$

Exercise:

Problem: $\frac{4}{x-2} \geq \frac{3}{x+1}$

Solution:

$[-10,-1) \cup (2,\infty)$

Exercise:

Problem: $\frac{5}{x-1} \leq \frac{4}{x+2}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation.

Exercise:

Problem:

Given the function $R(x) = \frac{x-5}{x-2}$, find the values of x that make the function less than or equal to 0.

Solution:

$(2,5]$

Exercise:

Problem:

Given the function $R(x) = \frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

Exercise:

Problem:

Given the function $R(x) = \frac{x-6}{x+2}$, find the values of x that make the function less than or equal to 0.

Solution:

$(-\infty,-2) \cup [6,\infty)$

Exercise:

Problem:

Given the function $R(x) = \frac{x+1}{x-4}$, find the values of x that make the function less than or equal to 0.

Writing Exercises

Exercise:

Problem: Write the steps you would use to explain solving rational inequalities to your little brother.

Solution:

Answers will vary.

Exercise:

Problem: Create a rational inequality whose solution is $(-\infty, -2] \cup [4, \infty)$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve rational inequalities.			
solve an inequality with rational functions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Chapter Review Exercises

Simplify, Multiply, and Divide Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

Exercise:

Problem: $\frac{5a+3}{3a-2}$

Solution:

$a \neq \frac{2}{3}$

Exercise:

Problem: $\frac{b-7}{b^2-25}$

Exercise:

Problem: $\frac{5x^2y^2}{8y}$

Solution:

$y \neq 0$

Exercise:

Problem: $\frac{x-3}{x^2-x-30}$

Simplify Rational Expressions

In the following exercises, simplify.

Exercise:

Problem: $\frac{18}{24}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{9m^4}{18mn^3}$

Exercise:

Problem: $\frac{x^2+7x+12}{x^2+8x+16}$

Solution:

$\frac{x+3}{x+4}$

Exercise:

Problem: $\frac{7v-35}{25-v^2}$

Multiply Rational Expressions

In the following exercises, multiply.

Exercise:

Problem: $\frac{5}{8} \cdot \frac{4}{15}$

Solution:

$\frac{1}{6}$

Exercise:

Problem: $\frac{3xy^2}{8y^3} \cdot \frac{16y^2}{24x}$

Exercise:

Problem: $\frac{72x-12x^2}{8x+32} \cdot \frac{x^2+10x+24}{x^2-36}$

Solution:

$\frac{-3x}{2}$

Exercise:

Problem: $\frac{6y^2-2y-10}{9-y^2} \cdot \frac{y^2-6y+9}{6y^2+29y-20}$

Divide Rational Expressions

In the following exercises, divide.

Exercise:

Problem: $\frac{x^2-4x+12}{x^2+8x+12} \div \frac{x^2-36}{3x}$

Solution:

$\frac{3x}{(x+6)(x+6)}$

Exercise:

Problem: $\frac{y^2-16}{4} \div \frac{y^3-64}{2y^2+8y+32}$

Exercise:

Problem: $\frac{11+w}{w-9} \div \frac{121-w^2}{9-w}$

Solution:

$\frac{1}{11+w}$

Exercise:

Problem: $\frac{3y^2-12y-63}{4y+3} \div (6y^2-42y)$

Exercise:

Problem: $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

Solution:

$\frac{5}{c+4}$

Exercise:

Problem: $\frac{8a^2+16a}{a-4} \cdot \frac{a^2+2a-24}{a^2+7a+10} \div \frac{2a^2-6a}{a+5}$

Multiply and Divide Rational Functions
Exercise:

Problem: Find $R\left(x\right)=f\left(x\right)\cdot g\left(x\right)$ where $f\left(x\right)=\frac{9x^2+9x}{x^2-3x-4}$ and $g\left(x\right)=\frac{x^2-16}{3x^2+12x}$.

Solution:

$R\left(x\right)=3$

Exercise:

Problem: Find $R\left(x\right)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{27x^2}{3x-21}$ and $g\left(x\right)=\frac{9x^2+54x}{x^2-x-42}$.

Add and Subtract Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, perform the indicated operations.
Exercise:

Problem: $\frac{7}{15}+\frac{8}{15}$

Solution:

1

Exercise:

Problem: $\frac{4a^2}{2a-1}-\frac{1}{2a-1}$

Exercise:

Problem: $\frac{y^2+10y}{y+5}+\frac{25}{y+5}$

Solution:

$y+5$

Exercise:

Problem: $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

Exercise:

Problem: $\frac{x^2}{x-7} - \frac{3x+28}{x-7}$

Solution:

$x + 4$

Exercise:

Problem: $\frac{y^2}{y+11} - \frac{121}{y+11}$

Exercise:

Problem: $\frac{4q^2-q+3}{q^2+6q+5} - \frac{3q^2-q-6}{q^2+6q+5}$

Solution:

$\frac{q-3}{q+5}$

Exercise:

Problem: $\frac{5t+4t+3}{t^2-25} - \frac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions Whose Denominators Are Opposites

In the following exercises, add and subtract.

Exercise:

Problem: $\frac{18w}{6w-1} + \frac{3w-2}{1-6w}$

Solution:

$\frac{15w+2}{6w-1}$

Exercise:

Problem: $\frac{a^2+3a}{a^2-4} - \frac{3a-8}{4-a^2}$

Exercise:

Problem: $\frac{2b^2+3b-15}{b^2-49} - \frac{b^2+16b-1}{49-b^2}$

Solution:

$\frac{3b-2}{b+7}$

Exercise:

Problem: $\frac{8y^2-10y+7}{2y-5} + \frac{2y^2+7y+2}{5-2y}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

Exercise:

Problem: $\frac{7}{a^2-3a-10}$, $\frac{3a}{a^2-a-20}$

Solution:

$(a+2)(a-5)(a+4)$

Exercise:

Problem: $\frac{6}{n^2-4}$, $\frac{2n}{n^2-4n+4}$

Exercise:

Problem: $\frac{5}{3p^2+17p-6}$, $\frac{2m}{3p^2-23p-8}$

Solution:

$(3p+1)(p+6)(p+8)$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

Exercise:

Problem: $\frac{7}{5a}+\frac{3}{2b}$

Exercise:

Problem: $\frac{2}{c-2}+\frac{9}{c+3}$

Solution:

$\frac{11c-12}{(c-2)(c+3)}$

Exercise:

Problem: $\frac{3x}{x^2-9}+\frac{5}{x^2+6x+9}$

Exercise:

Problem: $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

Solution:

$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$

Exercise:

Problem: $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

Exercise:

Problem: $\frac{3y}{y+2}-\frac{y+2}{y+8}$

Solution:

$\frac{2(y^2+10y-2)}{(y+2)(y+8)}$

Exercise:

Problem: $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

Exercise:

Problem: $\frac{7m+3}{m+2}-5$

Solution:

$\frac{2m-7}{m+2}$

Exercise:

Problem: $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

Exercise:

Problem: $\frac{8a}{a^2-64}-\frac{4}{a+8}$

Solution:

$\frac{4}{a-8}$

Exercise:

Problem: $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Add and Subtract Rational Functions

In the following exercises, find $R\left(x\right)=f\left(x\right)+g\left(x\right)$ where $f\left(x\right)$ and $g\left(x\right)$ are given.
Exercise:

Problem: $f(x)=\frac{2x^2+12x-11}{x^2+3x-10}$,$g\left(x\right)=\frac{x+1}{2-x}$

Solution:

$R\left(x\right)=\frac{x+8}{x+5}$

Exercise:

Problem: $f\left(x\right)=\frac{-4x+31}{x^2+x-30},g\left(x\right)=\frac{5}{x+6}$

In the following exercises, find $R\left(x\right)=f\left(x\right)-g\left(x\right)$ where $f\left(x\right)$ and $g\left(x\right)$ are given.

Exercise:

Problem: $f\left(x\right)=\frac{4x}{x^2-121},g\left(x\right)=\frac{2}{x-11}$

Solution:

$R\left(x\right)=\frac{2}{x+11}$

Exercise:

Problem: $f\left(x\right)=\frac{7}{x+6},g\left(x\right)=\frac{14x}{x^2-36}$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

Exercise:

Problem: $\frac{\frac{7x}{x+2}}{\frac{14x^2}{x^2-4}}$

Solution:

$\frac{x-2}{2x}$

Exercise:

Problem: $\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

Exercise:

Problem: $\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

Solution:

$\frac{(x-8)(x-5)}{2}$

Exercise:

Problem: $\frac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Exercise:

Problem: $\frac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

Solution:

$\frac{11}{8}$

Exercise:

Problem: $\frac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

Exercise:

Problem: $\frac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

Solution:

$\frac{z-5}{23z+21}$

Exercise:

Problem: $\frac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

7.4 Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

Exercise:

Problem: $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

Solution:

$x=\frac{6}{7}$

Exercise:

Problem: $1-\frac{2}{m}=\frac{8}{m^2}$

Exercise:

Problem: $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

Solution:

$b=\frac{3}{2}$

Exercise:

Problem: $\frac{3}{q+8}-\frac{2}{q-2}=1$

Exercise:

Problem: $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

Solution:

no solution

Exercise:

Problem: $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve Rational Equations that Involve Functions

Exercise:

Problem:

For rational function, $f\left(x\right)=\frac{x+2}{x^2-6x+8}$, ⓐ find the domain of the function ⓑ solve $f\left(x\right)=1$ ⓒ find the points on the graph at this function value.

Solution:

ⓐ The domain is all real numbers except $x\ne 2$ and $x\ne 4$. ⓑ $x=1, x=6$
ⓒ $(1,1),(6,1)$

Exercise:

Problem:

For rational function, $f\left(x\right)=\frac{2-x}{x^2+7x+10}$, ⓐ find the domain of the function ⓑ solve $f\left(x\right)=2$ ⓒ find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

Exercise:

Problem: $\frac{V}{l}=hw$ for l.

Solution:

$l=\frac{V}{hw}$

Exercise:

Problem: $\frac{1}{x}-\frac{2}{y}=5$ for y.

Exercise:

Problem: $x=\frac{y+5}{z-7}$ for z.

Solution:

$z=\frac{y+5+7x}{x}$

Exercise:

Problem: $P=\frac{k}{V}$ for V.

Solve Applications with Rational Equations

Solve Proportions

In the following exercises, solve.

Exercise:

Problem: $\frac{x}{4}=\frac{3}{5}$

Solution:

$\frac{12}{5}$

Exercise:

Problem: $\frac{3}{y}=\frac{9}{5}$

Exercise:

Problem: $\frac{s}{s+20}=\frac{3}{7}$

Solution:

15

Exercise:

Problem: $\frac{t-3}{5}=\frac{t+2}{9}$

Solve Using Proportions

In the following exercises, solve.
Exercise:

Problem:

Rachael had a 21-ounce strawberry shake that has 739 calories. How many calories are there in a 32-ounce shake?

Solution:

1161 calories

Exercise:

Problem:

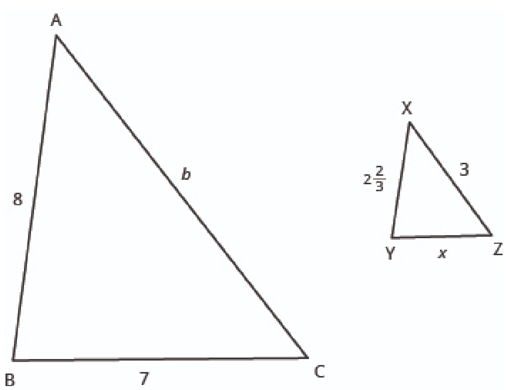
Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.
Exercise:

Problem:

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



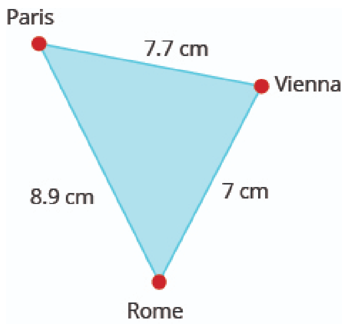
Solution:

$b = 9; x = 2\frac{1}{3}$

Exercise:

Problem:

On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from
ⓐ Paris to Rome
ⓑ Paris to Vienna



Exercise:

Problem:

Francesca is 5.75 feet tall. Late one afternoon, her shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Solution:

23 feet

Exercise:

Problem:

The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5-foot-tall Natasha cast a 1.1-foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise:

Problem:

When making the 5-hour drive home from visiting her parents, Lolo ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles when it turned bad. How fast did she drive when the weather was bad?

Solution:

45 mph

Exercise:

Problem:

Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

Exercise:

Problem:

Josue can ride his bicycle 8 mph faster than Arjun can ride his bike. It takes Luke 3 hours longer than Josue to ride 48 miles. How fast can John ride his bike?

Solution:

16 mph

Exercise:

Problem:

Curtis was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

Exercise:

Problem:

Brandy can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Solution:

$\frac{4}{5}$ hour

Exercise:

Problem:

Prem takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

Exercise:

Problem:

Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Solution:

12 days

Exercise:

Problem:

Marta and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Solve Direct Variation Problems

In the following exercises, solve.

Exercise:

Problem: If y varies directly as x when $y = 9$ and $x = 3$, find x when $y = 21$.

Solution:

7

Exercise:

Problem: If y varies inversely as x when $y = 20$ and $x = 2$, find y when $x = 4$.

Exercise:

Problem:

Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Solution:

301 mph

Exercise:

Problem:

If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

Exercise:

Problem:

The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solution:

288 feet

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise:

Problem: If m varies inversely with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2$.

Exercise:

Problem:

The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

Solution:

97 tickets

Exercise:

Problem:

On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

Solve Rational Inequalities

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

Exercise:

Problem: $\frac{x-3}{x+4} \leq 0$

Solution:

$(-4, 3]$

Exercise:

Problem: $\frac{5x}{x-2} > 1$

Exercise:

Problem: $\frac{3x-2}{x-4} \leq 2$

Solution:

$[-6, 4)$

Exercise:

Problem: $\frac{1}{x^2-4x-12} < 0$

Exercise:

Problem: $\frac{1}{2} - \frac{4}{x^2} \geq \frac{1}{x}$

Solution:

$(-\infty, -2] \cup [4, \infty)$

Exercise:

Problem: $\frac{4}{x-2} < \frac{3}{x+1}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation

Exercise:

Problem:

Given the function, $R\left(x\right)=\frac{x-5}{x-2}$, find the values of x that make the function greater than or equal to 0.

Solution:

$(-\infty ,2)\cup [5,\infty)$

Exercise:

Problem:

Given the function, $R\left(x\right)=\frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

Exercise:

Problem:

The function
$C(x)=150x+100{,}000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than \$160.

Solution:

ⓐ $c(x)=\frac{150x+100000}{x}$
ⓑ More than 10,000 items must be produced to keep the average cost below \$160 per item.

Exercise:

Problem:

Tillman is starting his own business by selling tacos at the beach. Accounting for the cost of his food truck and ingredients for the tacos, the function $C(x)=2x+6{,}000$ represents the cost for Tillman to produce x, tacos. Find ⓐ the average cost function, $c(x)$ for Tillman's Tacos ⓑ how many tacos should Tillman produce so that the average cost is less than \$4.

Practice Test

In the following exercises, simplify.
Exercise:

Problem: $\frac{4a^2b}{12ab^2}$

Solution:

$\frac{a}{3b}$

Exercise:

Problem: $\frac{6x-18}{x^2-9}$

In the following exercises, perform the indicated operation and simplify.

Exercise:

Problem: $\frac{4x}{x+2} \cdot \frac{x^2+5x+6}{12x^2}$

Solution:

$\frac{x+3}{3x}$

Exercise:

Problem: $\frac{2y^2}{y^2-1} \div \frac{y^3-y^2+y}{y^3-1}$

Exercise:

Problem: $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$

Solution:

$\frac{x-3}{x+9}$

Exercise:

Problem: $\frac{-3a}{3a-3} + \frac{5a}{a^2+3a-4}$

Exercise:

Problem: $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$

Solution:

$\frac{3n-2}{n-1}$

Exercise:

Problem: $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Exercise:

Problem: $\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

Solution:

$\frac{n-m}{m+n}$

In the following exercises, solve each equation.

Exercise:

Problem: $\frac{1}{x} + \frac{3}{4} = \frac{5}{8}$

Exercise:

Problem: $\frac{1}{z-5}+\frac{1}{z+5}=\frac{1}{z^2-25}$

Solution:

$z=\frac{1}{2}$

Exercise:

Problem: $\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$

In the following exercises, solve each rational inequality and write the solution in interval notation.

Exercise:

Problem: $\frac{6x}{x-6}\le 2$

Solution:

$[-3,6)$

Exercise:

Problem: $\frac{2x+3}{x-6}>1$

Exercise:

Problem: $\frac{1}{2}+\frac{12}{x^2}\ge\frac{5}{x}$

Solution:

$(-\infty,0)\cup(0,4]\cup[6,\infty)$

In the following exercises, find $R\left(x\right)$ given $f\left(x\right)=\frac{x-4}{x^2-3x-10}$ and $g\left(x\right)=\frac{x-5}{x^2-2x-8}$.

Exercise:

Problem: $R(x)=f(x)-g(x)$

Exercise:

Problem: $R(x)=f(x)\cdot g(x)$

Solution:

$R\left(x\right)=\frac{1}{(x+2)(x+2)}$

Exercise:

Problem: $R(x)=f(x)\div g(x)$

Exercise:

Problem: Given the function, $R\left(x\right)=\frac{2}{2x^2+x-15}$, find the values of x that make the function less than or equal to 0.

Solution:

$\left(2,5\right]$

In the following exercises, solve.

Exercise:

Problem: If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

Exercise:

Problem: If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

Solution:

$y=\frac{81}{16}$

Exercise:

Problem:

Matheus can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Matheus' speed on his bike?

Exercise:

Problem:

Oliver can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Oliver's dad working alone to split the logs?

Solution:

Oliver's dad would take $4\frac{4}{5}$ hours to split the logs himself.

Exercise:

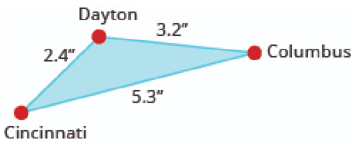
Problem:

The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

Exercise:

Problem:

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio. The diagram gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Solution:

The distance between Dayton and Columbus is 64 miles.

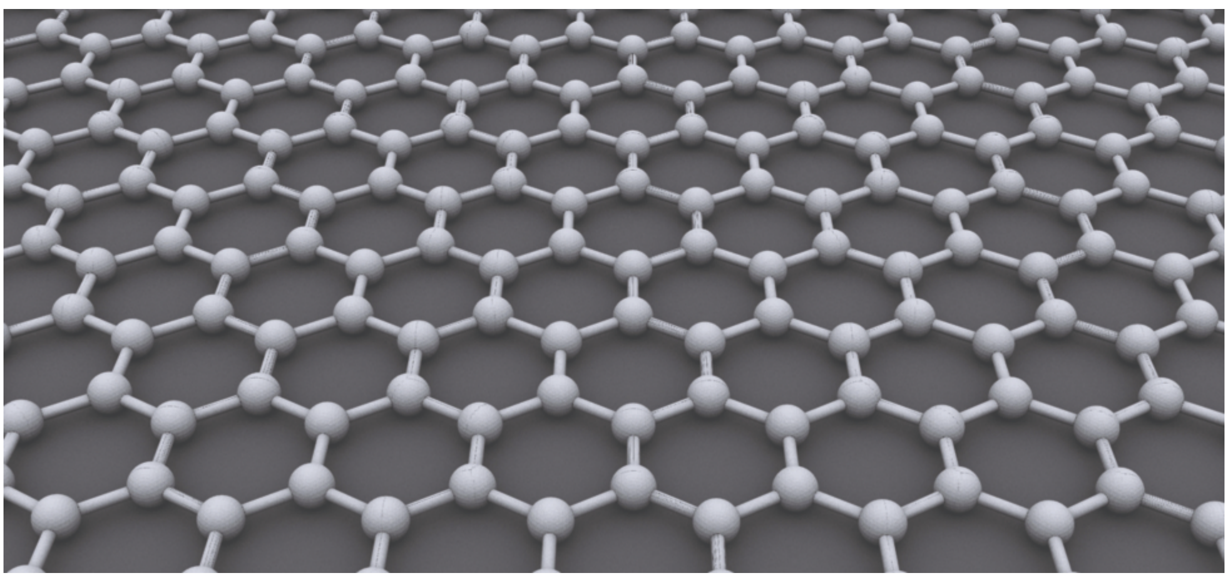
Glossary

critical point of a rational inequality
: The critical point of a rational inequality is a number which makes the rational expression zero or undefined.

rational inequality
: A rational inequality is an inequality that contains a rational expression.

Introduction
class="introduction"

Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: “AlexanderAIUS” / Wikimedia Commons)



Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon

atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

Simplify Expressions with Roots
By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Note:
Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)^2$ ⓑ -9^2 ⓒ $(-9)^3$.
   If you missed this problem, review [link].
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review [link].
3. Simplify: ⓐ $x^3 \cdot x^3$ ⓑ $y^2 \cdot y^2 \cdot y^2$ ⓒ $z^3 \cdot z^3 \cdot z^3 \cdot z^3$.
   If you missed this problem, review [link].

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write n^2 and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

Equation:

$$\begin{array}{c} 13^2 \text{ is read "13 squared"} \\ 169 \text{ is called the } \textit{square} \text{ of } 13, \text{ since } 13^2 = 169 \\ 13 \text{ is a } \textit{square root} \text{ of } 169 \end{array}$$

Note:
Square and Square Root of a number

Square

Equation:

$$\text{If } n^2 = m, \text{ then } m \text{ is the } \textbf{square} \text{ of } n.$$

Square Root

Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a } \textbf{square root} \text{ of } m.$$

Notice $(-13)^2 = 169$ also, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Note:
Square Root Notation
Equation:

$$\sqrt{m} \text{ is read “the square root of } m\text{”}.$$
$$\text{If } n^2 = m, \text{ then } n = \sqrt{m}, \text{ for } n \geq 0.$$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{289}$.

Solution:

ⓐ

	$\sqrt{144}$
Since $12^2 = 144$.	12

ⓑ

	$-\sqrt{289}$
Since $17^2 = 289$ and the negative is in front of the radical sign.	-17

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{225}$.

Solution:

ⓐ -8 ⓑ 15

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{100}$ ⓑ $-\sqrt{121}$.

Solution:

ⓐ 10 ⓑ -11

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?
Equation:

$$(\ )^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{64}$.

Solution:

ⓐ

	$\sqrt{-196}$
There is no real number whose square is -196.	$\sqrt{-196}$ is not a real number.

ⓑ

	$-\sqrt{64}$
The negative is in front of the radical.	-8

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{81}$.

Solution:

ⓐ not a real number ⓑ -9

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

Solution:

ⓐ -7 ⓑ not a real number

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from −5 to 5. See [link].

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of −2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

Note:
n^{th} Root of a Number
Equation:

$$\begin{array}{c}\text{If } b^n = a, \text{ then } b \text{ is an } n^{th} \text{ root of } a.\\ \text{The principal } n^{th} \text{ root of } a \text{ is written } \sqrt[n]{a}.\\ n \text{ is called the } \textbf{index} \text{ of the radical.}\end{array}$$

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to [link] to help find higher roots.
Equation:

$$\begin{array}{rclcrcl} 4^3 & = & 64 & \qquad & \sqrt[3]{64} & = & 4 \\ 3^4 & = & 81 & & \sqrt[4]{81} & = & 3 \\ (-2)^5 & = & -32 & & \sqrt[5]{-32} & = & -2 \end{array}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Note:
Properties of $\sqrt[n]{a}$
When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

We will apply these properties in the next two examples.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{64}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution:

ⓐ

	$\sqrt[3]{64}$
Since $4^3 = 64$.	4

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

Solution:

ⓐ 3 ⓑ 4 ⓒ 3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$.

Solution:

ⓐ 10 ⓑ 2 ⓒ 3

In this example be alert for the negative signs as well as even and odd powers.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{-243}$.

Solution:

ⓐ

	$\sqrt[3]{-125}$
Since $(-5)^3 = -125$.	-5

ⓑ

	$\sqrt[4]{-16}$
Think, $(?)^4 = -16.$ No real number raised to the fourth power is negative.	Not a real number.

ⓒ

	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-256}$ ⓒ $\sqrt[5]{-32}$.

Solution:

ⓐ -3 ⓑ not real ⓒ -2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Solution:

ⓐ -6 ⓑ not real ⓒ -4

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4}=2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

Number	Square Root
4	2
9	3
16	4
25	5

11 $\quad\sqrt{11}$

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

Number	Cube Root
8	2
27	3
64	4
125	5

91 $\quad\sqrt[3]{91}$

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

Example:
Exercise:

Problem: Estimate each root between two consecutive whole numbers: ⓐ $\sqrt{105}$ ⓑ $\sqrt[3]{43}$.

Solution:

ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

<table>
<tr><td></td><td>$\sqrt{105}$</td></tr>
<tr><td></td><td>
<table>
<tr><th></th><th>Number</th><th>Square Root</th><th></th></tr>
<tr><td></td><td>81</td><td>9</td><td></td></tr>
<tr><td></td><td>100</td><td>10</td><td></td></tr>
<tr><td>105</td><td>121</td><td>11</td><td>$\sqrt{105}$</td></tr>
<tr><td></td><td>144</td><td>12</td><td></td></tr>
</table>
</td></tr>
<tr><td>Locate 105 between two consecutive perfect squares.</td><td>$100 < 105 < 121$</td></tr>
<tr><td>$\sqrt{105}$ is between their square roots.</td><td>$10 < \sqrt{105} < 11$</td></tr>
</table>

ⓑ Similarly we locate 43 between two perfect cube numbers.

<table>
<tr><td></td><td>$\sqrt[3]{43}$</td></tr>
<tr><td></td><td>
<table>
<tr><th></th><th>Number</th><th>Cube Root</th><th></th></tr>
<tr><td></td><td>8</td><td>2</td><td></td></tr>
<tr><td></td><td>27</td><td>3</td><td></td></tr>
<tr><td>43</td><td>64</td><td>4</td><td>$\sqrt[3]{43}$</td></tr>
<tr><td></td><td>125</td><td>5</td><td></td></tr>
</table>
</td></tr>
<tr><td>Locate 43 between two consecutive perfect cubes.</td><td>$27 < 43 < 64$</td></tr>
<tr><td>$\sqrt[3]{43}$ is between their cube roots.</td><td>$3 < \sqrt[3]{43} < 4$</td></tr>
</table>

Note:

Exercise:

Problem: Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{38}$ ⓑ $\sqrt[3]{93}$

Solution:

ⓐ $6 < \sqrt{38} < 7$
ⓑ $4 < \sqrt[3]{93} < 5$

Note:
Exercise:

Problem: Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{84}$ ⓑ $\sqrt[3]{152}$

Solution:

ⓐ $9 < \sqrt{84} < 10$
ⓑ $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that
Equation:

$$\begin{array}{c} \sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24 \\ \sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11 \end{array}$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:
Equation:

$$\begin{array}{rclcrcl} (2.236067978)^2 & = & 5.000000002 & \qquad & (3.105422799)^4 & = & 92.999999991 \\ (2.24)^2 & = & 5.0176 & & (3.11)^4 & = & 93.54951841 \end{array}$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Example:
Exercise:

Problem: Round to two decimal places: ⓐ $\sqrt{17}$ ⓑ $\sqrt[3]{49}$ ⓒ $\sqrt[4]{51}$.

Solution:

ⓐ

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626\ldots$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

ⓑ

	$\sqrt[3]{49}$
Use the calculator $\sqrt[y]{x}$ key.	$3.659305710\ldots$
Round to two decimal places.	3.66
	$\sqrt[3]{49} \approx 3.66$

ⓒ

	$\sqrt[4]{51}$
Use the calculator $\sqrt[y]{x}$ key.	$2.6723451177\ldots$
Round to two decimal places.	2.67
	$\sqrt[4]{51} \approx 2.67$

Note:
Exercise:

Problem: Round to two decimal places:

ⓐ $\sqrt{11}$ ⓑ $\sqrt[3]{71}$ ⓒ $\sqrt[4]{127}$.

Solution:

ⓐ ≈ 3.32 ⓑ ≈ 4.14
ⓒ ≈ 3.36

Note:
Exercise:

Problem: Round to two decimal places:

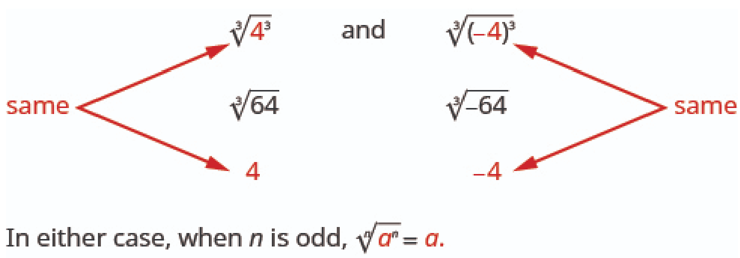
ⓐ $\sqrt{13}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{98}$.

Solution:

ⓐ ≈ 3.61 ⓑ ≈ 4.38
ⓒ ≈ 3.15

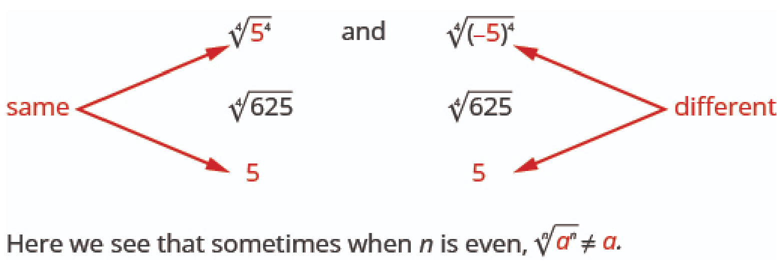
Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,



But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Note:
Simplifying Odd and Even Roots
For any integer $n \geq 2$,
Equation:

when the index n is odd	$\sqrt[n]{a^n} = a$
when the index n is even	$\sqrt[n]{a^n} = \|a\|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution:

ⓐ We use the absolute value to be sure to get the positive root.

	$\sqrt{x^2}$
Since the index n is even, $\sqrt[n]{a^n} = \|a\|$.	$\|x\|$

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

	$\sqrt[3]{m^3}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	m

ⓒ

	$\sqrt[4]{p^4}$
Since the index n is even $\sqrt[n]{a^n} = \|a\|$.	$\|p\|$

ⓓ

	$\sqrt[5]{y^5}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	y

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|b|$ ⓑ w ⓒ $|m|$ ⓓ q

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|y|$ ⓑ p ⓒ $|z|$ ⓓ q

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

Equation:

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

	$\sqrt{a^{2m}}$
Since $(a^m)^2 = a^{2m}$.	$\sqrt{(a^m)^2}$
Since n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$\lvert a^m \rvert$
	So $\sqrt{a^{2m}} = \lvert a^m \rvert$.

We apply this concept in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution:

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	$\sqrt{\left(x^3\right)^2}$
Since the index n is even $\sqrt{a^n} = \lvert a \rvert$.	$\left\lvert x^3 \right\rvert$

ⓑ

Since $\left(y^8\right)^2 = y^{16}$.

Since the index n is even $\sqrt[n]{a^n} = |a|$.
In this case the absolute value sign is not needed as y^8 is positive.

$$\sqrt{y^{16}}$$
$$\sqrt{\left(y^8\right)^2}$$
$$y^8$$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^{18}}$ ⓑ $\sqrt{z^{12}}$.

Solution:

ⓐ $\left|y^9\right|$ ⓑ z^6

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

Solution:

ⓐ m^2 ⓑ $\left|b^5\right|$

The next example uses the same idea for highter roots.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution:

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
Since n is odd, $\sqrt[n]{a^n} = a$.	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

Solution:

ⓐ $\left|u^3\right|$ ⓑ v^5

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$

Solution:

ⓐ c^4 ⓑ d^4

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = \left|a^m\right|$ works in much the same way.

Equation:

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.

Solution:

ⓐ

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$\sqrt{(4n)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$4\lvert n \rvert$

ⓑ

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-\sqrt{(9c)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$-9\lvert c \rvert$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.

Solution:

ⓐ $8\lvert x \rvert$ ⓑ $-10\lvert p \rvert$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.

Solution:

ⓐ $13\lvert y \rvert$ ⓑ $-11\lvert y \rvert$

This example just takes the idea farther as it has roots of higher index.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution:

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{\left(2q^3\right)^4}$
Take the fourth root.	$2\left\|q^3\right\|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

Solution:

ⓐ $3x^9$ ⓑ $3\left|q^7\right|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{125q^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

Solution:

ⓐ $5p^3$ ⓑ $3q^5$

The next examples have two variables.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{36x^2y^2}$ ⓑ $\sqrt{121a^6b^8}$ ⓒ $\sqrt[3]{64p^{63}q^9}$.

Solution:

ⓐ

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$	$\sqrt{(6xy)^2}$
Take the square root.	$6\,\lvert xy\rvert$

ⓑ

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$	$\sqrt{\left(11a^3b^4\right)^2}$
Take the square root.	$11\,\lvert a^3\rvert b^4$

ⓒ

	$\sqrt[3]{64p^{63}q^9}$
Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$	$\sqrt[3]{\left(4p^{21}q^3\right)^3}$
Take the cube root.	$4p^{21}q^3$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{100a^2b^2}$ ⓑ $\sqrt{144p^{12}q^{20}}$ ⓒ $\sqrt[3]{8x^{30}y^{12}}$

Solution:

ⓐ $10\,\lvert ab\rvert$ ⓑ $12p^6q^{10}$
ⓒ $2x^{10}y^4$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{225m^2n^2}$ ⓑ $\sqrt{169x^{10}y^{14}}$ ⓒ $\sqrt[3]{27w^{36}z^{15}}$

Solution:

ⓐ $15\,|mn|$ ⓑ $13\,\left|x^5y^7\right|$
ⓒ $3w^{12}z^5$

Note:
Access this online resource for additional instruction and practice with simplifying expressions with roots.

- [Simplifying Variables Exponents with Roots using Absolute Values]

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of *m*'
  - If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

  - The square root of *m*, $\sqrt{m}$, is a positive number whose square is *m*.
- **n^{th} Root of a Number**
  - If $b^n = a$, then *b* is an n^{th} root of *a*.
  - The principal n^{th} root of *a* is written $\sqrt[n]{a}$.
  - *n* is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When *n* is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When *n* is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Simplifying Odd and Even Roots**
  - For any integer $n \geq 2$,
    - when *n* is odd $\sqrt[n]{a^n} = a$
    - when *n* is even $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt{64}$ ⓑ $-\sqrt{81}$

Solution:

ⓐ 8 ⓑ -9

Exercise:

Problem: ⓐ $\sqrt{169}$ ⓑ $-\sqrt{100}$

Exercise:

Problem: ⓐ $\sqrt{196}$ ⓑ $-\sqrt{1}$

Solution:

ⓐ 14 ⓑ -1

Exercise:

Problem: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{121}$

Exercise:

Problem: ⓐ $\sqrt{\frac{4}{9}}$ ⓑ $-\sqrt{0.01}$

Solution:

ⓐ $\frac{2}{3}$ ⓑ -0.1

Exercise:

Problem: ⓐ $\sqrt{\frac{64}{121}}$ ⓑ $-\sqrt{0.16}$

Exercise:

Problem: ⓐ $\sqrt{-121}$ ⓑ $-\sqrt{289}$

Solution:

ⓐ not real number ⓑ -17

Exercise:

Problem: ⓐ $-\sqrt{400}$ ⓑ $\sqrt{-36}$

Exercise:

Problem: ⓐ $-\sqrt{225}$ ⓑ $\sqrt{-9}$

Solution:

ⓐ -15 ⓑ not real number

Exercise:

Problem: ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{256}$

Exercise:

Problem: ⓐ $\sqrt[3]{216}$ ⓑ $\sqrt[4]{256}$

Solution:

ⓐ 6 ⓑ 4

Exercise:

Problem: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$

Exercise:

Problem: ⓐ $\sqrt[3]{512}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{1}$

Solution:

ⓐ 8 ⓑ 3 ⓒ 1

Exercise:

Problem: ⓐ $\sqrt[3]{125}$ ⓑ $\sqrt[4]{1296}$ ⓒ $\sqrt[5]{1024}$

Exercise:

Problem: ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-32}$

Solution:

ⓐ -2 ⓑ not real ⓒ -2

Exercise:

ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
Problem: ⓒ $\sqrt[5]{-243}$

Exercise:

ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
Problem: ⓒ $\sqrt[5]{-1024}$

Solution:

ⓐ -5 ⓑ not real ⓒ -4

Exercise:

ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
Problem: ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

Exercise:

Problem: ⓐ $\sqrt{70}$ ⓑ $\sqrt[3]{71}$

Solution:

ⓐ $8 < \sqrt{70} < 9$
ⓑ $4 < \sqrt[3]{71} < 5$

Exercise:

Problem: ⓐ $\sqrt{55}$ ⓑ $\sqrt[3]{119}$

Exercise:

Problem: ⓐ $\sqrt{200}$ ⓑ $\sqrt[3]{137}$

Solution:

ⓐ $14 < \sqrt{200} < 15$
ⓑ $5 < \sqrt[3]{137} < 6$

Exercise:

Problem: ⓐ $\sqrt{172}$ ⓑ $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

Exercise:

Problem: ⓐ $\sqrt{19}$ ⓑ $\sqrt[3]{89}$ ⓒ $\sqrt[4]{97}$

Solution:

ⓐ 4.36 ⓑ ≈ 4.46
ⓒ ≈ 3.14

Exercise:

Problem: ⓐ $\sqrt{21}$ ⓑ $\sqrt[3]{93}$ ⓒ $\sqrt[4]{101}$

Exercise:

Problem: ⓐ $\sqrt{53}$ ⓑ $\sqrt[3]{147}$ ⓒ $\sqrt[4]{452}$

Solution:

ⓐ 7.28 ⓑ ≈ 5.28
ⓒ ≈ 4.61

Exercise:

Problem: ⓐ $\sqrt{47}$ ⓑ $\sqrt[3]{163}$ ⓒ $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

Exercise:

Problem: ⓐ $\sqrt[5]{u^5}$ ⓑ $\sqrt[8]{v^8}$

Solution:

ⓐ u ⓑ $|v|$

Exercise:

Problem: ⓐ $\sqrt[3]{a^3}$ ⓑ $\sqrt[9]{b^9}$

Exercise:

Problem: ⓐ $\sqrt[4]{y^4}$ ⓑ $\sqrt[7]{m^7}$

Solution:

ⓐ $|y|$ ⓑ m

Exercise:

Problem: ⓐ $\sqrt[8]{k^8}$ ⓑ $\sqrt[6]{p^6}$

Exercise:

Problem: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$

Solution:

ⓐ $|x^3|$ ⓑ y^8

Exercise:

Problem: ⓐ $\sqrt{a^{14}}$ ⓑ $\sqrt{w^{24}}$

Exercise:

Problem: ⓐ $\sqrt{x^{24}}$ ⓑ $\sqrt{y^{22}}$

Solution:

ⓐ x^{12} ⓑ $|y^{11}|$

Exercise:

Problem: ⓐ $\sqrt{a^{12}}$ ⓑ $\sqrt{b^{26}}$

Exercise:

Problem: ⓐ $\sqrt[3]{x^9}$ ⓑ $\sqrt[4]{y^{12}}$

Solution:

ⓐ x^3 ⓑ $|y^3|$

Exercise:

Problem: ⓐ $\sqrt[5]{a^{10}}$ ⓑ $\sqrt[3]{b^{27}}$

Exercise:

Problem: ⓐ $\sqrt[4]{m^8}$ ⓑ $\sqrt[5]{n^{20}}$

Solution:

ⓐ m^2 ⓑ n^4

Exercise:

Problem: ⓐ $\sqrt[6]{r^{12}}$ ⓑ $\sqrt[3]{s^{30}}$

Exercise:

Problem: ⓐ $\sqrt{49x^2}$ ⓑ $-\sqrt{81x^{18}}$

Solution:

ⓐ $7\,|x|$ ⓑ $-9\,|x^9|$

Exercise:

Problem: ⓐ $\sqrt{100y^2}$ ⓑ $-\sqrt{100m^{32}}$

Exercise:

Problem: ⓐ $\sqrt{121m^{20}}$ ⓑ $-\sqrt{64a^2}$

Solution:

ⓐ $11m^{10}$ ⓑ $-8\,|a|$

Exercise:

Problem: ⓐ $\sqrt{81x^{36}}$
ⓑ $-\sqrt{25x^2}$

Exercise:

Problem: ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

Solution:

ⓐ $2x^2$ ⓑ $2y^2$

Exercise:

Problem: ⓐ $\sqrt[3]{-8c^9}$ ⓑ $\sqrt[3]{125d^{15}}$

Exercise:

Problem: ⓐ $\sqrt[3]{216a^6}$ ⓑ $\sqrt[5]{32b^{20}}$

Solution:

ⓐ $6a^2$ ⓑ $2b^4$

Exercise:

Problem: ⓐ $\sqrt[7]{128r^{14}}$ ⓑ $\sqrt[4]{81s^{24}}$

Exercise:

Problem: ⓐ $\sqrt{144x^2y^2}$ ⓑ $\sqrt{169w^8y^{10}}$ ⓒ $\sqrt[3]{8a^{51}b^6}$

Solution:

ⓐ $12\left|xy\right|$ ⓑ $13w^4\left|y^5\right|$
ⓒ $2a^{17}b^2$

Exercise:

Problem: ⓐ $\sqrt{196a^2b^2}$ ⓑ $\sqrt{81p^{24}q^6}$ ⓒ $\sqrt[3]{27p^{45}q^9}$

Exercise:

Problem: ⓐ $\sqrt{121a^2b^2}$ ⓑ $\sqrt{9c^8d^{12}}$ ⓒ $\sqrt[3]{64x^{15}y^{66}}$

Solution:

ⓐ $11\,|ab|$ ⓑ $3c^4d^6$
ⓒ $4x^5y^{22}$

Exercise:

Problem: ⓐ $\sqrt{225x^2y^2z^2}$
ⓑ $\sqrt{36r^6s^{20}}$
ⓒ $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

Exercise:

Problem: Why is there no real number equal to $\sqrt{-64}$?

Solution:

Answers will vary.

Exercise:

Problem: What is the difference between 9^2 and $\sqrt{9}$?

Exercise:

Problem: Explain what is meant by the n^{th} root of a number.

Solution:

Answers will vary.

Exercise:

Problem:

Explain the difference of finding the n^{th} root of a number when the index is even compared to when the index is odd.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

square of a number
: If $n^2 = m$, then m is the square of n.

square root of a number
: If $n^2 = m$, then n is a square root of m.

Simplify Radical Expressions
By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{y^3}{y^{11}}$.
   If you missed this problem, review [link].
3. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review [link].

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Note:
Simplified Radical Expression
For real numbers a and m, and $n \geq 2$,
Equation:

$$\sqrt[n]{a} \text{ is considered simplified if } a \text{ has no factors of } m^n$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that $(ab)^n = a^n b^n$. The corresponding of **Product Property of Roots** says that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Note:
Product Property of n^{th} Roots
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then
Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example:
Simplify Square Roots Using the Product Property of Roots
Exercise:

Problem: Simplify: $\sqrt{98}$.

Solution:

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.		$7\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{48}$.

Solution:

$4\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{45}$.

Solution:

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Note:
Simplify a radical expression using the Product Property.

Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
Use the product rule to rewrite the radical as the product of two radicals.
Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

Solution:

ⓐ

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

ⓑ

	$\sqrt[3]{16}$
Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$	$\sqrt[3]{8 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓒ

	$\sqrt[4]{243}$
Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$	$\sqrt[4]{81 \cdot 3}$
Rewrite the radical as the product of two radicals	$\sqrt[4]{81} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{288}$ ⓑ $\sqrt[3]{81}$ ⓒ $\sqrt[4]{64}$.

Solution:

ⓐ $12\sqrt{2}$ ⓑ $3\sqrt[3]{3}$ ⓒ $2\sqrt[4]{4}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{432}$ ⓑ $\sqrt[3]{625}$ ⓒ $\sqrt[4]{729}$.

Solution:

ⓐ $12\sqrt{3}$ ⓑ $5\sqrt[3]{5}$ ⓒ $3\sqrt[4]{9}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^3}$ ⓑ $\sqrt[3]{x^4}$ ⓒ $\sqrt[4]{x^7}$.

Solution:

ⓐ

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$\lvert x \rvert \sqrt{x}$

ⓑ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}.$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x \sqrt[3]{x}$

ⓒ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x \rvert \sqrt[4]{x^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

Solution:

ⓐ $b^2\sqrt{b}$ ⓑ $|y|\sqrt[4]{y^2}$ ⓒ $z\sqrt[3]{z^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{p^9}$ ⓑ $\sqrt[5]{y^8}$ ⓒ $\sqrt[6]{q^{13}}$

Solution:

ⓐ $p^4\sqrt{p}$ ⓑ $p\sqrt[5]{p^3}$
ⓒ $q^2\sqrt[6]{q}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{72n^7}$ ⓑ $\sqrt[3]{24x^7}$ ⓒ $\sqrt[4]{80y^{14}}$.

Solution:

ⓐ

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

ⓑ

$$\sqrt[3]{24x^7}$$

Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{8x^6 \cdot 3x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$$

Rewrite the first radicand as $\left(2x^2\right)^3$.

$$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$$

Simplify.

$$2x^2\sqrt[3]{3x}$$

ⓒ

$$\sqrt[4]{80y^{14}}$$

Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16y^{12} \cdot 5y^2}$$

Rewrite the radical as the product of two radicals.

$$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$$

Rewrite the first radicand as $\left(2y^3\right)^4$.

$$\sqrt[4]{\left(2y^3\right)^4} \cdot \sqrt[4]{5y^2}$$

Simplify.

$$2\left|y^3\right|\sqrt[4]{5y^2}$$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{32y^5}$ ⓑ $\sqrt[3]{54p^{10}}$ ⓒ $\sqrt[4]{64q^{10}}$.

Solution:

ⓐ $4y^2\sqrt{2y}$ ⓑ $3p^3\sqrt[3]{2p}$
ⓒ $2q^2\sqrt[4]{4q^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{75a^9}$ ⓑ $\sqrt[3]{128m^{11}}$ ⓒ $\sqrt[4]{162n^7}$.

Solution:

ⓐ $5a^4\sqrt{3a}$ ⓑ $4m^3\sqrt[3]{2m^2}$
ⓒ $3\left|n\right|\sqrt[4]{2n^3}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{63u^3v^5}$ ⓑ $\sqrt[3]{40x^4y^5}$ ⓒ $\sqrt[4]{48x^4y^7}$.

Solution:

ⓐ

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4\cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4}\cdot\sqrt{7uv}$
Rewrite the first radicand as $\left(3uv^2\right)^2$.	$\sqrt{\left(3uv^2\right)^2}\cdot\sqrt{7uv}$
Simplify.	$3\left|u\right|v^2\sqrt{7uv}$

ⓑ

$$\sqrt[3]{40x^4y^5}$$

Rewrite the radicand as a product using the largest perfect cube factor. $\sqrt[3]{8x^3y^3 \cdot 5xy^2}$

Rewrite the radical as the product of two radicals. $\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$

Rewrite the first radicand as $(2xy)^3$. $\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$

Simplify. $2xy\sqrt[3]{5xy^2}$

ⓒ

$$\sqrt[4]{48x^4y^7}$$

Rewrite the radicand as a product using the largest perfect fourth power factor. $\sqrt[4]{16x^4y^4 \cdot 3y^3}$

Rewrite the radical as the product of two radicals. $\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$

Rewrite the first radicand as $(2xy)^4$. $\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$

Simplify. $2|xy|\sqrt[4]{3y^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{98a^7b^5}$ ⓑ $\sqrt[3]{56x^5y^4}$ ⓒ $\sqrt[4]{32x^5y^8}$.

Solution:

ⓐ $7|a^3|b^2\sqrt{2ab}$
ⓑ $2xy\sqrt[3]{7x^2y}$ ⓒ $2|x|y^2\sqrt[4]{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{180m^9n^{11}}$ ⓑ $\sqrt[3]{72x^6y^5}$ ⓒ $\sqrt[4]{80x^7y^4}$.

Solution:

ⓐ $6m^4\left|n^5\right|\sqrt{5mn}$
ⓑ $2x^2y\sqrt[3]{9y^2}$ ⓒ $2\left|xy\right|\sqrt[4]{5x^3}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution:

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4=-16$.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-81}$.

Solution:

ⓐ -4 ⓑ no real number

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

Solution:

ⓐ $-5\sqrt[3]{5}$ ⓑ no real number

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example:
Exercise:

Problem: Simplify: ⓐ $3+\sqrt{32}$ ⓑ $\frac{4-\sqrt{48}}{2}$.

Solution:

ⓐ

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

	$\frac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\frac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\frac{4\left(1-\sqrt{3}\right)}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\frac{\cancel{2}\cdot 2\left(1-\sqrt{3}\right)}{\cancel{2}}$
Simplify.	$2\left(1-\sqrt{3}\right)$

Note:
Exercise:

Problem: Simplify: ⓐ $5+\sqrt{75}$ ⓑ $\frac{10-\sqrt{75}}{5}$

Solution:

ⓐ $5+5\sqrt{3}$ ⓑ $2-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $2+\sqrt{98}$ ⓑ $\frac{6-\sqrt{45}}{3}$

Solution:

ⓐ $2+7\sqrt{2}$ ⓑ $2-\sqrt{5}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{45}{80}}$ ⓑ $\sqrt[3]{\frac{16}{54}}$ ⓒ $\sqrt[4]{\frac{5}{80}}$.

Solution:

ⓐ

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
Simplify. Note $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.	$\frac{3}{4}$

ⓑ

	$\sqrt[3]{\frac{16}{54}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[3]{\frac{2 \cdot 8}{2 \cdot 27}}$
Simplify the fraction by removing common factors.	$\sqrt[3]{\frac{8}{27}}$
Simplify. Note $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$.	$\frac{2}{3}$

ⓒ

	$\sqrt[4]{\frac{5}{80}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[4]{\frac{5\cdot 1}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt[4]{\frac{1}{16}}$
Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.	$\frac{1}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{75}{48}}$ ⓑ $\sqrt[3]{\frac{54}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$.

Solution:

ⓐ $\frac{5}{4}$ ⓑ $\frac{3}{5}$ ⓒ $\frac{2}{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{98}{162}}$ ⓑ $\sqrt[3]{\frac{24}{375}}$ ⓒ $\sqrt[4]{\frac{4}{324}}$.

Solution:

ⓐ $\frac{7}{9}$ ⓑ $\frac{2}{5}$ ⓒ $\frac{1}{3}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,
Equation:

...

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{m^6}{m^4}}$ ⓑ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓒ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution:

ⓐ

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	$\|m\|$

ⓑ

	$\sqrt[3]{\frac{a^8}{a^5}}$
Use the Quotient Property of exponents to simplify the fraction under the radical first.	$\sqrt[3]{a^3}$
Simplify.	a

ⓒ

	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Use the Quotient Property of exponents to simplify the fraction under the radical first.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{(a^2)^4}$
Simplify.	a^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{a^8}{a^6}}$ ⓑ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓒ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

Solution:

ⓐ $|a|$ ⓑ $|x|$ ⓒ y^3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{x^{14}}{x^{10}}}$ ⓑ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓒ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Solution:

ⓐ x^2 ⓑ m^2 ⓒ n^2

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.
Equation:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

Note:
Quotient Property of Radical Expressions
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
Equation:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example:
How to Simplify the Quotient of Radical Expressions
Exercise:

Problem: Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution:

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{24p^3}{49}}$.

Solution:

$\frac{2|p|\sqrt{6p}}{7}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{48x^5}{100}}$.

Solution:

$\frac{2x^2\sqrt{3x}}{5}$

Note:
Simplify a square root using the Quotient Property.

Simplify the fraction in the radicand, if possible.
Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Simplify the radicals in the numerator and the denominator.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{45x^5}{y^4}}$ ⓑ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓒ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Solution:

ⓐ

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

ⓑ

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6\cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3}\cdot\sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

ⓒ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{80m^3}{n^6}}$ ⓑ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓒ $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

Solution:

ⓐ $\frac{4|m|\sqrt{5m}}{|n^3|}$ ⓑ $\frac{3c^3\sqrt[3]{4c}}{d^2}$
ⓒ $\frac{2x^2\sqrt[4]{5x^2}}{|y|}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{54u^7}{v^8}}$ ⓑ $\sqrt[3]{\frac{40r^3}{s^6}}$ ⓒ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Solution:

ⓐ $\frac{3u^3\sqrt{6u}}{v^4}$ ⓑ $\frac{2r\sqrt[3]{5}}{s^2}$

ⓒ $\frac{3|m^3|\sqrt[4]{2m^2}}{|n^3|}$

Be sure to simplify the fraction in the radicand first, if possible.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{18p^5q^7}{32pq^2}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

Solution:

ⓐ

	$\sqrt{\frac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\frac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\frac{3p^2q^2\sqrt{q}}{4}$

ⓑ

	$\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[3]{\dfrac{8x^3y^5}{27}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$
Simplify.	$\dfrac{2xy\sqrt[3]{y^2}}{3}$

ⓒ

	$\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[4]{\dfrac{a^5b^4}{16}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$
Simplify.	$\dfrac{\lvert ab\rvert\sqrt[4]{a}}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\dfrac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$.

Solution:

ⓐ $\dfrac{5\lvert y\rvert\sqrt{x}}{6}$ ⓑ $\dfrac{2xy\sqrt[3]{y^2}}{3}$
ⓒ $\dfrac{\lvert ab\rvert\sqrt[4]{a}}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$ ⓑ $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$ ⓒ $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$.

Solution:

ⓐ $\frac{2|m|\sqrt{3}}{5|n^3|}$ ⓑ $\frac{3xy\sqrt[3]{x^2}}{5}$
ⓒ $\frac{2|ab|\sqrt[4]{a^2}}{3}$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{48a^7}}{\sqrt{3a}}$ ⓑ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution:

ⓐ

	$\frac{\sqrt{48a^7}}{\sqrt{3a}}$
The denominator cannot be simplified, so use the Quotient Property to write as one radical.	$\sqrt{\frac{48a^7}{3a}}$
Simplify the fraction under the radical.	$\sqrt{16a^6}$
Simplify.	$4\left\|a^3\right\|$

ⓑ

$$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓒ

$$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.	$\sqrt[4]{\frac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\|x\|\sqrt[4]{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{98z^5}}{\sqrt{2z}}$ ⓑ $\frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

Solution:

ⓐ $7z^2$ ⓑ $-5\sqrt[3]{2}$
ⓒ $3|m|\sqrt[4]{2m^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{128m^9}}{\sqrt{2m}}$ ⓑ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓒ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

Solution:

ⓐ $8m^4$ ⓑ -4 ⓒ $3\,|n|\sqrt[4]{2}$

Note:
Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n
- **Product Property of n^th Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
    $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
- **How to simplify a radical expression using the Product Property**

  Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
  Use the product rule to rewrite the radical as the product of two radicals.
  Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **How to simplify a radical expression using the Quotient Property.**

  Simplify the fraction in the radicand, if possible.
  Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.
Exercise:

Problem: $\sqrt{27}$

Solution:

$3\sqrt{3}$

Exercise:

Problem: $\sqrt{80}$

Exercise:

Problem: $\sqrt{125}$

Solution:

$5\sqrt{5}$

Exercise:

Problem: $\sqrt{96}$

Exercise:

Problem: $\sqrt{147}$

Solution:

$7\sqrt{3}$

Exercise:

Problem: $\sqrt{450}$

Exercise:

Problem: $\sqrt{800}$

Solution:

$20\sqrt{2}$

Exercise:

Problem: $\sqrt{675}$

Exercise:

Problem: ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

Solution:

ⓐ $2\sqrt[4]{2}$ ⓑ $2\sqrt[5]{2}$

Exercise:

Problem: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

Exercise:

Problem: ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

Solution:

ⓐ $2\sqrt[5]{2}$ ⓑ $4\sqrt[3]{4}$

Exercise:

Problem: ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

In the following exercises, simplify using absolute value signs as needed.

Exercise:

ⓐ $\sqrt{y^{11}}$
ⓑ $\sqrt[3]{r^5}$
Problem: ⓒ $\sqrt[4]{s^{10}}$

Solution:

ⓐ $|y^5|\sqrt{y}$ ⓑ $r\sqrt[3]{r^2}$ ⓒ $s^2\sqrt[4]{s^2}$

Exercise:

ⓐ $\sqrt{m^{13}}$
ⓑ $\sqrt[5]{u^7}$
Problem: ⓒ $\sqrt[6]{v^{11}}$

Exercise:

ⓐ $\sqrt{n^{21}}$
ⓑ $\sqrt[3]{q^8}$
Problem: ⓒ $\sqrt[8]{n^{10}}$

Solution:

ⓐ $n^{10}\sqrt{n}$ ⓑ $q^2\sqrt[3]{q^2}$
ⓒ $|n|\sqrt[8]{n^2}$

Exercise:

ⓐ $\sqrt{r^{25}}$
ⓑ $\sqrt[5]{p^8}$
Problem: ⓒ $\sqrt[4]{m^5}$

Exercise:

ⓐ $\sqrt{125r^{13}}$
ⓑ $\sqrt[3]{108x^5}$
Problem: ⓒ $\sqrt[4]{48y^6}$

Solution:

ⓐ $5r^6\sqrt{5r}$ ⓑ $3x\sqrt[3]{4x^2}$
ⓒ $2\left|y\right|\sqrt[4]{3y^2}$

Exercise:

ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
Problem: ⓒ $\sqrt[6]{128b^7}$

Exercise:

ⓐ $\sqrt{242m^{23}}$
ⓑ $\sqrt[4]{405m10}$
Problem: ⓒ $\sqrt[5]{160n^8}$

Solution:

ⓐ $11\left|m^{11}\right|\sqrt{2m}$ ⓑ $3m^2\sqrt[4]{5m^2}$ ⓒ $2n\sqrt[5]{5n^3}$

Exercise:

ⓐ $\sqrt{175n^{13}}$
ⓑ $\sqrt[5]{512p^5}$
Problem: ⓒ $\sqrt[4]{324q^7}$

Exercise:

ⓐ $\sqrt{147m^7n^{11}}$
ⓑ $\sqrt[3]{48x^6y^7}$
Problem: ⓒ $\sqrt[4]{32x^5y^4}$

Solution:

ⓐ $7\left|m^3n^5\right|\sqrt{3mn}$ ⓑ $2x^2y^2\sqrt[3]{6y}$ ⓒ $2\left|xy\right|\sqrt[4]{2x}$

Exercise:

ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
Problem: ⓒ $\sqrt[4]{80x^8y^9}$

Exercise:

ⓐ $\sqrt{192q^3r^7}$
ⓑ $\sqrt[3]{54m^9n^{10}}$
Problem: ⓒ $\sqrt[4]{81a^9b^8}$

Solution:

ⓐ $8\left|qr^3\right|\sqrt{3qr}$ ⓑ $3m^3n^3\sqrt[3]{2n}$ ⓒ $3a^2b^2\sqrt[4]{a}$

Exercise:

ⓐ $\sqrt{150m^9n^3}$
ⓑ $\sqrt[3]{81p^7q^8}$
Problem: ⓒ $\sqrt[4]{162c^{11}d^{12}}$

Exercise:

ⓐ $\sqrt[3]{-864}$
Problem: ⓑ $\sqrt[4]{-256}$

Solution:

ⓐ $-6\sqrt[3]{4}$ ⓑ not real

Exercise:

ⓐ $\sqrt[5]{-486}$
Problem: ⓑ $\sqrt[6]{-64}$

Exercise:

ⓐ $\sqrt[5]{-32}$
Problem: ⓑ $\sqrt[8]{-1}$

Solution:

ⓐ -2 ⓑ not real

Exercise:

ⓐ $\sqrt[3]{-8}$
Problem: ⓑ $\sqrt[4]{-16}$

Exercise:

Problem: ⓐ $5+\sqrt{12}$ ⓑ $\frac{10-\sqrt{24}}{2}$

Solution:

ⓐ $5+2\sqrt{3}$ ⓑ $5-\sqrt{6}$

Exercise:

Problem: ⓐ $8+\sqrt{96}$ ⓑ $\frac{8-\sqrt{80}}{4}$

Exercise:

Problem: ⓐ $1+\sqrt{45}$ ⓑ $\frac{3+\sqrt{90}}{3}$

Solution:

ⓐ $1+3\sqrt{5}$ ⓑ $1+\sqrt{10}$

Exercise:

Problem: ⓐ $3+\sqrt{125}$ ⓑ $\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

Exercise:

Problem: ⓐ $\sqrt{\frac{45}{80}}$ ⓑ $\sqrt[3]{\frac{8}{27}}$ ⓒ $\sqrt[4]{\frac{1}{81}}$

Solution:

ⓐ $\frac{3}{4}$ ⓑ $\frac{2}{3}$ ⓒ $\frac{1}{3}$

Exercise:

Problem: ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{100}{36}}$ ⓑ $\sqrt[3]{\frac{81}{375}}$ ⓒ $\sqrt[4]{\frac{1}{256}}$

Solution:

ⓐ $\frac{5}{3}$ ⓑ $\frac{3}{5}$ ⓒ $\frac{1}{4}$

Exercise:

Problem: ⓐ $\sqrt{\frac{121}{16}}$ ⓑ $\sqrt[3]{\frac{16}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{x^{10}}{x^6}}$ ⓑ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓒ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

Solution:

ⓐ x^2 ⓑ p^3 ⓒ $|q|$

Exercise:

Problem: ⓐ $\sqrt{\frac{p^{20}}{p^{10}}}$ ⓑ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓒ $\sqrt[8]{\frac{m^{12}}{m^4}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

Solution:

ⓐ $\frac{1}{y^2}$ ⓑ u^2 ⓒ $|v^3|$

Exercise:

Problem: ⓐ $\sqrt{\frac{q^8}{q^{14}}}$ ⓑ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓒ $\sqrt[4]{\frac{c^{21}}{c^9}}$

Exercise:

Problem: $\sqrt{\frac{96x^7}{121}}$

Solution:

$\frac{4|x^3|\sqrt{6x}}{11}$

Exercise:

Problem: $\sqrt{\frac{108y^4}{49}}$

Exercise:

Problem: $\sqrt{\frac{300m^5}{64}}$

Solution:

$\frac{10m^2\sqrt{3m}}{8}$

Exercise:

Problem: $\sqrt{\frac{125n^7}{169}}$

Exercise:

Problem: $\sqrt{\frac{98r^5}{100}}$

Solution:

$\frac{7r^2\sqrt{2r}}{10}$

Exercise:

Problem: $\sqrt{\frac{180s^{10}}{144}}$

Exercise:

Problem: $\sqrt{\frac{28q^6}{225}}$

Solution:

$\frac{2|q^3|\sqrt{7}}{15}$

Exercise:

Problem: $\sqrt{\frac{150r^3}{256}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{75r^9}{s^8}}$ ⓑ $\sqrt[3]{\frac{54a^8}{b^3}}$ ⓒ $\sqrt[4]{\frac{64c^5}{d^4}}$

Solution:

ⓐ $\frac{5r^4\sqrt{3r}}{s^4}$ ⓑ $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$ ⓒ $\frac{2|c|\sqrt[4]{4c}}{|d|}$

Exercise:

Problem: ⓐ $\sqrt{\frac{72x^5}{y^6}}$ ⓑ $\sqrt[5]{\frac{96r^{11}}{s^5}}$ ⓒ $\sqrt[6]{\frac{128u^7}{v^{12}}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

Solution:

ⓐ $\frac{2|p^3|\sqrt{7p}}{|q|}$ ⓑ $\frac{3s^2\sqrt[3]{3s^2}}{t}$
ⓒ $\frac{2|p^3|\sqrt[4]{4p^3}}{|q^3|}$

Exercise:

Problem: ⓐ $\sqrt{\frac{45r^3}{s^{10}}}$
ⓑ $\sqrt[3]{\frac{625u^{10}}{v^3}}$
ⓒ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{32x^5y^3}{18x^3y}}$
ⓑ $\sqrt[3]{\frac{5x^6y^9}{40x^5y^3}}$
ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

Solution:

ⓐ $\frac{4|xy|}{3}$ ⓑ $\frac{y^2\sqrt[3]{x}}{2}$ ⓒ $\frac{|ab|\sqrt[4]{a}}{4}$

Exercise:

Problem: ⓐ $\sqrt{\frac{75r^6s^8}{48rs^4}}$ ⓑ $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$ ⓒ $\sqrt[4]{\frac{32m^9n^2}{162mn^2}}$

Exercise:

Problem: ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$ ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$ ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

Solution:

ⓐ $\frac{1}{2|pq|}$ ⓑ $\frac{2cd\sqrt[5]{2d^2}}{5}$
ⓒ $\frac{|mn|\sqrt[6]{2}}{2}$

Exercise:

Problem: ⓐ $\sqrt{\frac{50r^5s^2}{128r^2s^6}}$ ⓑ $\sqrt[3]{\frac{24m^9n^7}{375m^4n}}$ ⓒ $\sqrt[4]{\frac{81m^2n^8}{256m^1n^2}}$

Exercise:

Problem: ⓐ $\frac{\sqrt{45p^9}}{\sqrt{5q^2}}$ ⓑ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$ ⓒ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

Solution:

ⓐ $\frac{3p^4\sqrt{p}}{|q|}$ ⓑ $2\sqrt[4]{2}$
ⓒ $2x\sqrt[5]{2x}$

Exercise:

Problem:
ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$
ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$
ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Exercise:

Problem:
ⓐ $\frac{\sqrt{50m^7}}{\sqrt{2m}}$
ⓑ $\sqrt[3]{\frac{1250}{2}}$
ⓒ $\sqrt[4]{\frac{486y^9}{2y^3}}$

Solution:

ⓐ $5\left|m^3\right|$ ⓑ $5\sqrt[3]{5}$
ⓒ $3\left|y\right|\sqrt[4]{3y^2}$

Exercise:

Problem:
ⓐ $\frac{\sqrt{72n^{11}}}{\sqrt{2n}}$
ⓑ $\sqrt[3]{\frac{162}{6}}$
ⓒ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Writing Exercises

Exercise:

Problem: Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

Exercise:

Problem: Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Simplify Rational Exponents
By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Note:
Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review [link].
2. Simplify: $\left(4x^2y^5\right)^3$.
   If you missed this problem, review [link].
3. Simplify: 5^{-3}.
   If you missed this problem, review [link].

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of *p*.

	$(8^p)^3 = 8$
Multiply the exponents on the left.	$8^{3p} = 8$
Write the exponent 1 on the right.	$8^{3p} = 8^1$
Since the bases are the same, the exponents must be equal.	$3p = 1$
Solve for p.	$p = \frac{1}{3}$

So $\left(8^{\frac{1}{3}}\right)^3 = 8$. But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Note:
Rational Exponent $a^{\frac{1}{n}}$
If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then
Equation:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example:
Exercise:

Problem: Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution:

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

Note:
Exercise:

Problem: Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt{t}$ ⓑ $\sqrt[3]{m}$ ⓒ $\sqrt[4]{r}$

Note:
Exercise:

Problem: Write as a radial expression: ⓐ $b^{\frac{1}{6}}$ ⓑ $z^{\frac{1}{5}}$ ⓒ $p^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt[6]{b}$ ⓑ $\sqrt[5]{z}$ ⓒ $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution:

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2. Put parentheses around the entire expression $5y$.	$(5y)^{\frac{1}{2}}$

ⓑ

The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.

$$\sqrt[3]{4x}$$

$$(4x)^{\frac{1}{3}}$$

ⓒ

The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.

$$3\sqrt[4]{5z}$$

$$3(5z)^{\frac{1}{4}}$$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

Solution:

ⓐ $(10m)^{\frac{1}{2}}$ ⓑ $(3n)^{\frac{1}{5}}$
ⓒ $3(6y)^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

Solution:

ⓐ $(3k)^{\frac{1}{7}}$ ⓑ $(5j)^{\frac{1}{4}}$
ⓒ $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example:

Exercise:

Problem: Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution:

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

Note:
Exercise:

Problem: Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

Solution:

ⓐ 6 ⓑ 2 ⓒ 2

Note:
Exercise:

Problem: Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Solution:

ⓐ 10 ⓑ 3 ⓒ 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example:
Exercise:

Problem: Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution:

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
	$\sqrt[4]{(-2)^4}$
Simplify.	No real solution.

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fouth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-64)^{-\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

Solution:

ⓐ No real solution ⓑ -8
ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Solution:

ⓐ No real solution ⓑ -4
ⓒ $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

Equation:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

Note:
Rational Exponent $a^{\frac{m}{n}}$
For any positive integers *m* and *n*,
Equation:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

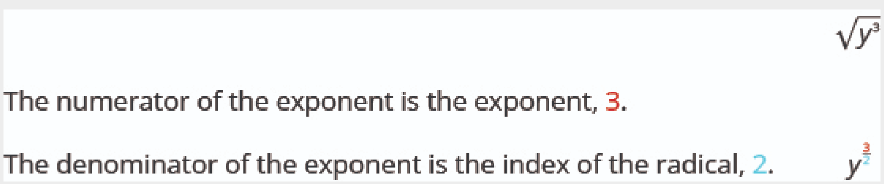
Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\left(\sqrt[3]{2x}\right)^4$ ⓒ $\sqrt{\left(\frac{3a}{4b}\right)^3}$.

Solution:

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ



ⓑ

$\left(\sqrt[3]{2x}\right)^4$

The numerator of the exponent is the exponent, 4.

The denominator of the exponent is the index of the radical, 3. $(2x)^{\frac{4}{3}}$

ⓒ

$\sqrt{\left(\frac{3a}{4b}\right)^3}$

The numerator of the exponent is the exponent, 3.

The denominator of the exponent is the index of the radical, 2. $\left(\frac{3a}{4b}\right)^{\frac{3}{2}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\left(\sqrt[4]{3y}\right)^3$ ⓒ $\sqrt{\left(\frac{2m}{3n}\right)^5}$.

Solution:

ⓐ $x^{\frac{5}{2}}$ ⓑ $(3y)^{\frac{3}{4}}$ ⓒ $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\left(\sqrt[3]{5ab}\right)^5$ ⓒ $\sqrt{\left(\frac{7xy}{z}\right)^3}$.

Solution:

ⓐ $a^{\frac{2}{5}}$ ⓑ $(5ab)^{\frac{5}{3}}$
ⓒ $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

Example:
Exercise:

Problem: Simplify: ⓐ $125^{\frac{2}{3}}$ ⓑ $16^{-\frac{3}{2}}$ ⓒ $32^{-\frac{2}{5}}$.

Solution:

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

ⓑ We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

$$16^{-\frac{3}{2}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{\left(\sqrt{16}\right)^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

ⓒ

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n} = \frac{1}{a^n}$.

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{\left(\sqrt[5]{32}\right)^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

Note:
Exercise:

Problem: Simplify: ⓐ $27^{\frac{2}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

Solution:

ⓐ 9 ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

Solution:

ⓐ 8 ⓑ $\frac{1}{9}$ ⓒ $\frac{1}{125}$

Example:
Exercise:

Problem: Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution:

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-\left(\sqrt{25}\right)^3$
Simplify the radical.	$-(5)^3$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{\left(\sqrt{25}\right)^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$\left(\sqrt{-25}\right)^3$
There is no real number whose square root is -25.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

Solution:

ⓐ -64 ⓑ $-\frac{1}{64}$ ⓒ not a real number

Note:
Exercise:

Problem: Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Solution:

ⓐ -729 ⓑ $-\frac{1}{729}$ ⓒ not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Note:
Properties of Exponents
If *a* and *b* are real numbers and *m* and *n* are rational numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \\ \textbf{Quotient Property} & \frac{a^m}{a^n} & = & a^{m-n},\ a \neq 0 \\ \textbf{Zero Exponent Definition} & a^0 & = & 1,\ a \neq 0 \\ \textbf{Quotient to a Power Property} & \left(\frac{a}{b}\right)^m & = & \frac{a^m}{b^m},\ b \neq 0 \\ \textbf{Negative Exponent Property} & a^{-n} & = & \frac{1}{a^n},\ a \neq 0 \end{array}$$

We will apply these properties in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ ⓑ $\left(z^9\right)^{\frac{2}{3}}$ ⓒ $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

Solution:

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

	$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$
The bases are the same, so we add the exponents.	$x^{\frac{1}{2}+\frac{5}{6}}$
Add the fractions.	$x^{\frac{8}{6}}$
Simplify the exponent.	$x^{\frac{4}{3}}$

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9\cdot\frac{2}{3}}$
Simplify.	z^6

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

	$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
	$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$
Simplify.	$\frac{1}{x^{\frac{4}{3}}}$

Note:
Exercise:

Problem: Simplify: ⓐ $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ ⓑ $\left(x^6\right)^{\frac{4}{3}}$ ⓒ $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

Solution:

ⓐ $x^{\frac{3}{2}}$ ⓑ x^8 ⓒ $\frac{1}{x}$

Note:
Exercise:

Problem: Simplify: ⓐ $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$ ⓑ $\left(m^9\right)^{\frac{2}{9}}$ ⓒ $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Solution:

ⓐ $y^{\frac{11}{8}}$ ⓑ m^2 ⓒ $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$.

Solution:

ⓐ

$$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

First we use the Product to a Power Property.

$$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

Rewrite 27 as a power of 3.

$$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiply the exponents.

$$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$$

Simplify.

$$9u^{\frac{1}{3}}$$

ⓑ

$$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

First we use the Product to a Power Property.

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

To raise a power to a power, we multiply the exponents.

$$mn^{\frac{3}{4}}$$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$.

Solution:

ⓐ $8x^{\frac{1}{5}}$ ⓑ $x^{\frac{1}{2}}y^{\frac{1}{3}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$ ⓑ $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$.

Solution:

ⓐ $729n^{\frac{3}{5}}$ ⓑ $a^2b^{\frac{2}{3}}$

We will use both the Product Property and the Quotient Property in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\frac{16\, x^{\frac{4}{3}} y^{-\frac{5}{6}}}{x^{-\frac{2}{3}} y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

Solution:

ⓐ

	$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$
Use the Product Property in the numerator, add the exponents.	$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$
Use the Quotient Property, subtract the exponents.	$x^{\frac{8}{4}}$
Simplify.	x^2

ⓑ Follow the order of operations to simplify inside the parenthese first.

	$\left(\frac{16\, x^{\frac{4}{3}} y^{-\frac{5}{6}}}{x^{-\frac{2}{3}} y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$
Use the Quotient Property, subtract the exponents.	$\left(\frac{16 x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$
Simplify.	$\left(\frac{16 x^2}{y}\right)^{\frac{1}{2}}$
Use the Product to a Power Property, multiply the exponents.	$\frac{4x}{y^{\frac{1}{2}}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\left(\frac{25 m^{\frac{1}{6}} n^{\frac{11}{6}}}{m^{\frac{2}{3}} n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$.

Solution:

ⓐ m^2 ⓑ $\frac{5n}{m^{\frac{1}{4}}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\left(\frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$.

Solution:

ⓐ u^3 ⓑ $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

Note:
Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent** $a^{\frac{1}{n}}$
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent** $a^{\frac{m}{n}}$
  - For any positive integers *m* and *n*,
    $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If *a*, *b* are real numbers and *m*, *n* are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $\left(a^m\right)^n = a^{m \cdot n}$

- **Product to a Power** $(ab)^m = a^m b^m$
- **Quotient Property** $\frac{a^m}{a^n} = a^{m-n},\ a \neq 0$
- **Zero Exponent Definition** $a^0 = 1,\ a \neq 0$
- **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$
- **Negative Exponent Property** $a^{-n} = \frac{1}{a^n},\ a \neq 0$

Practice Makes Perfect

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise:

Problem: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$

Solution:

ⓐ $\sqrt{x}$ ⓑ $\sqrt[3]{y}$ ⓒ $\sqrt[4]{z}$

Exercise:

Problem: ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

Exercise:

Problem: ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{9}}$ ⓒ $w^{\frac{1}{20}}$

Solution:

ⓐ $\sqrt[5]{u}$ ⓑ $\sqrt[9]{v}$ ⓒ $\sqrt[20]{w}$

Exercise:

Problem: ⓐ $g^{\frac{1}{7}}$ ⓑ $h^{\frac{1}{5}}$ ⓒ $j^{\frac{1}{25}}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: ⓐ $\sqrt[7]{x}$ ⓑ $\sqrt[9]{y}$ ⓒ $\sqrt[5]{f}$

Solution:

ⓐ $\frac{1}{x^7}$ ⓑ $\frac{1}{y^9}$ ⓒ $f^{\frac{1}{5}}$

Exercise:

Problem: ⓐ $\sqrt[8]{r}$ ⓑ $\sqrt[10]{s}$ ⓒ $\sqrt[4]{t}$

Exercise:

Problem: ⓐ $\sqrt[3]{7c}$ ⓑ $\sqrt[7]{12d}$ ⓒ $2\sqrt[4]{6b}$

Solution:

ⓐ $(7c)^{\frac{1}{4}}$ ⓑ $(12d)^{\frac{1}{7}}$
ⓒ $2(6b)^{\frac{1}{4}}$

Exercise:

Problem: ⓐ $\sqrt[4]{5x}$ ⓑ $\sqrt[8]{9y}$ ⓒ $7\sqrt[5]{3z}$

Exercise:

Problem: ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

Solution:

ⓐ $(21p)^{\frac{1}{2}}$ ⓑ $(8q)^{\frac{1}{4}}$
ⓒ $4(36r)^{\frac{1}{6}}$

Exercise:

Problem: ⓐ $\sqrt[3]{25a}$ ⓑ $\sqrt{3b}$ ⓒ $\sqrt[8]{40c}$

In the following exercises, simplify.

Exercise:

ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
Problem: ⓒ $64^{\frac{1}{2}}$

Solution:

ⓐ 9 ⓑ 5 ⓒ 8

Exercise:

ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
Problem: ⓒ $32^{\frac{1}{5}}$

Exercise:

ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
Problem: ⓒ $625^{\frac{1}{4}}$

Solution:

ⓐ 2 ⓑ 4 ⓒ 5

Exercise:

ⓐ $64^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
Problem: ⓒ $81^{\frac{1}{4}}$

Exercise:

ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
Problem: ⓒ $(216)^{-\frac{1}{3}}$

Solution:

ⓐ -6 ⓑ -6 ⓒ $\frac{1}{6}$

Exercise:

ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
Problem: ⓒ $(1000)^{-\frac{1}{3}}$

Exercise:

Problem:
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

Solution:

ⓐ not real ⓑ -3 ⓒ $\frac{1}{3}$

Exercise:

Problem:
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

Exercise:

Problem:
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

Solution:

ⓐ not real ⓑ -6 ⓒ $\frac{1}{6}$

Exercise:

Problem:
ⓐ $(-16)^{\frac{1}{4}}$
ⓑ $-16^{\frac{1}{4}}$
ⓒ $16^{-\frac{1}{4}}$

Exercise:

Problem:
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

Solution:

ⓐ not real ⓑ -10 ⓒ $\frac{1}{10}$

Exercise:

Problem:
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise:

Problem:
ⓐ $\sqrt{m^5}$
ⓑ $\left(\sqrt[3]{3y}\right)^7$
ⓒ $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

Solution:

ⓐ $m^{\frac{5}{2}}$ ⓑ $(3y)^{\frac{7}{3}}$ ⓒ $\left(\frac{4x}{5y}\right)^{\frac{3}{5}}$

Exercise:

Problem:
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

Exercise:

Problem:
ⓐ $\sqrt[5]{u^2}$
ⓑ $\left(\sqrt[3]{6x}\right)^5$
ⓒ $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

Solution:

ⓐ $u^{\frac{2}{5}}$ ⓑ $(6x)^{\frac{5}{3}}$ ⓒ $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

Exercise:

Problem: ⓐ $\sqrt[3]{a}$
ⓑ $\left(\sqrt[4]{21v}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

In the following exercises, simplify.

Exercise:

Problem: ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{\frac{-3}{2}}$
ⓒ $(-27)^{\frac{2}{3}}$

Solution:

ⓐ 32,768 ⓑ $\frac{1}{729}$ ⓒ 9

Exercise:

Problem: ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

Exercise:

Problem: ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $(-25)^{\frac{1}{2}}$

Solution:

ⓐ 4 ⓑ $\frac{1}{9}$ ⓒ not real

Exercise:

ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
Problem: ⓒ $(-100)^{\frac{3}{2}}$

Exercise:

ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
Problem: ⓒ $(-9)^{\frac{3}{2}}$

Solution:

ⓐ -27 ⓑ $-\frac{1}{27}$ ⓒ not real

Exercise:

ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
Problem: ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.
Exercise:

ⓐ $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$
ⓑ $\left(p^{12}\right)^{\frac{3}{4}}$
Problem: ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

Solution:

ⓐ $c^{\frac{7}{8}}$ ⓑ p^{9} ⓒ $\frac{1}{r}$

Exercise:

Problem: ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Exercise:

Problem: ⓐ $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$
ⓑ $\left(x^{12}\right)^{\frac{2}{3}}$
ⓒ $\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

Solution:

ⓐ $y^{\frac{5}{4}}$ ⓑ x^8 ⓒ $\frac{1}{m}$

Exercise:

Problem: ⓐ $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$
ⓑ $\left(h^6\right)^{\frac{4}{3}}$
ⓒ $\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

Exercise:

Problem: ⓐ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$
ⓑ $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Solution:

ⓐ $81q^2$ ⓑ $a^{\frac{1}{2}}b$

Exercise:

Problem: ⓐ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$
ⓑ $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$

Exercise:

Problem: ⓐ $\left(16\,u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ ⓑ $\left(4\,p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

Solution:

ⓐ $8u^{\frac{1}{4}}$ ⓑ $8p^{\frac{1}{2}}q^{\frac{3}{4}}$

Exercise:

Problem: ⓐ $\left(625\,n^{\frac{8}{3}}\right)^{\frac{3}{4}}$ ⓑ $\left(9\,x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

Exercise:

Problem: ⓐ $\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ ⓑ $\left(\frac{36\,s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

Solution:

ⓐ $r^{\frac{7}{2}}$ ⓑ $\frac{6s}{t}$

Exercise:

Problem: ⓐ $\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ ⓑ $\left(\frac{27\,b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Exercise:

ⓐ $\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

Problem: ⓑ $\left(\frac{8\,x^{\frac{5}{3}}\,y^{-\frac{1}{2}}}{27\,x^{-\frac{4}{3}}\,y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

Solution:

ⓐ c^2 ⓑ $\frac{2x}{3y}$

Exercise:

ⓐ $\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

Problem: ⓑ $\left(\frac{16\,m^{\frac{1}{5}}\,n^{\frac{3}{2}}}{81\,m^{\frac{9}{5}}\,n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Writing Exercises

Exercise:

Problem: Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

Solution:

Answers will vary.

Exercise:

Problem: Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Add, Subtract, and Multiply Radical Expressions
By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Note:
Before you get started, take this readiness quiz.

1. Add: $3x^2 + 9x - 5 - \left(x^2 - 2x + 3\right)$.
   If you missed this problem, review [link].
2. Simplify: $(2 + a)\,(4 - a)$.
   If you missed this problem, review [link].
3. Simplify: $(9 - 5y)^2$.
   If you missed this problem, review [link].

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Note:
Like Radicals
Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

Example:
Exercise:

Problem: Simplify: ⓐ $2\sqrt{2} - 7\sqrt{2}$ ⓑ $5\sqrt[3]{y} + 4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x} - 2\sqrt[4]{y}$.

Solution:

ⓐ

$$2\sqrt{2} - 7\sqrt{2}$$

Since the radicals are like, we subtract the coefficients.

$$-5\sqrt{2}$$

ⓑ

$$5\sqrt[3]{y} + 4\sqrt[3]{y}$$

Since the radicals are like, we add the coefficients.

$$9\sqrt[3]{y}$$

ⓒ

$$7\sqrt[4]{x} - 2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

Note:
Exercise:

Problem: Simplify: ⓐ $8\sqrt{2} - 9\sqrt{2}$ ⓑ $4\sqrt[3]{x} + 7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

Solution:

ⓐ $-\sqrt{2}$ ⓑ $11\sqrt[3]{x}$
ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $5\sqrt{3} - 9\sqrt{3}$ ⓑ $5\sqrt[3]{y} + 3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m} - 2\sqrt[3]{m}$.

Solution:

ⓐ $-4\sqrt{3}$ ⓑ $8\sqrt[3]{y}$
ⓒ $5\sqrt[4]{m} - 2\sqrt[3]{m}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands

are the same.

Example:
Exercise:

Problem: Simplify: ⓐ $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$.

Solution:

ⓐ

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

ⓑ

	$\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$
Since the radicals are like, we combine them.	$2\sqrt[4]{3xy}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy}$.

Solution:

ⓐ $-2\sqrt{7x}$ ⓑ $-\sqrt[4]{5xy}$

Note:
Exercise:

Problem: Simplify: ⓐ $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn} + \sqrt[3]{7mn} - 4\sqrt[3]{7mn}$.

Solution:

ⓐ $-\sqrt{3y}$ ⓑ $3\sqrt[3]{7mn}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{20}+3\sqrt{5}$ ⓑ $\sqrt[3]{24}-\sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$.

Solution:

ⓐ

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
	$2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

ⓑ

	$\sqrt[3]{24}-\sqrt[3]{375}$
Simplify the radicals.	$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$
	$2\sqrt[3]{3}-5\sqrt[3]{3}$
Combine the like radicals.	$-3\sqrt[3]{3}$

ⓒ

	$\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$
Simplify the radicals.	$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3}-\frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$
	$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3}-\frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$
	$\sqrt[4]{3}-2\sqrt[4]{3}$
Combine the like radicals.	$-\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{18}+6\sqrt{2}$ ⓑ $6\sqrt[3]{16}-2\sqrt[3]{250}$ ⓒ $\frac{2}{3}\sqrt[3]{81}-\frac{1}{2}\sqrt[3]{24}$.

Solution:

ⓐ $9\sqrt{2}$ ⓑ $2\sqrt[3]{2}$ ⓒ $\sqrt[3]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{27}+4\sqrt{3}$ ⓑ $4\sqrt[3]{5}-7\sqrt[3]{40}$ ⓒ $\frac{1}{2}\sqrt[3]{128}-\frac{5}{3}\sqrt[3]{54}$.

Solution:

ⓐ $7\sqrt{3}$ ⓑ $-10\sqrt[3]{5}$ ⓒ $-3\sqrt[3]{2}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example:
Exercise:

Problem: Simplify: ⓐ $9\sqrt{50m^2}-6\sqrt{48m^2}$ ⓑ $\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$.

Solution:

ⓐ

	$9\sqrt{50m^2}-6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$
	$9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}$
	$45m\sqrt{2}-24m\sqrt{3}$
The radicals are not like and so cannot be combined.	

ⓑ

	$\sqrt[3]{54n^5}-\sqrt[3]{16n^5}$
Simplify the radicals.	$\sqrt[3]{27n^3}\cdot\sqrt[3]{2n^2}-\sqrt[3]{8n^3}\cdot\sqrt[3]{2n^2}$
	$3n\sqrt[3]{2n^2}-2n\sqrt[3]{2n^2}$
Combine the like radicals.	$n\sqrt[3]{2n^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{32m^7} - \sqrt{50m^7}$ ⓑ $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$.

Solution:

ⓐ $-m^3\sqrt{2m}$ ⓑ $x^2\sqrt[3]{5x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{27p^3} - \sqrt{48p^3}$ ⓑ $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$.

Solution:

ⓐ $-p\sqrt{3p}$
ⓑ $4y\sqrt[3]{4y^2} - 2n\sqrt[3]{4n^2}$

Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Note:
Product Property of Roots
For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$ ⓑ $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$.

Solution:

ⓐ

	$\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4}\cdot\sqrt{5}$
Simplify.	$18\cdot 2\cdot\sqrt{5}$
	$36\sqrt{5}$

ⓑ

	$\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$
Multiply using the Product Property.	$20\sqrt[3]{24}$
Simplify the radical.	$20\sqrt[3]{8}\cdot\sqrt[3]{3}$
Simplify.	$20\cdot 2\cdot\sqrt[3]{3}$
	$40\sqrt[3]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(3\sqrt{2}\right)\left(2\sqrt{30}\right)$ ⓑ $\left(2\sqrt[3]{18}\right)\left(-3\sqrt[3]{6}\right)$.

Solution:

ⓐ $12\sqrt{15}$ ⓑ $-18\sqrt[3]{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(3\sqrt{3}\right)\left(3\sqrt{6}\right)$ ⓑ $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{6}\right)$.

Solution:

ⓐ $27\sqrt{2}$ ⓑ $-36\sqrt[3]{2}$

We follow the same procedures when there are variables in the radicands.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$ ⓑ $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$.

Solution:

ⓐ

	$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
Multiply.	$40\sqrt{18p^4}$
Simplify the radical.	$40\sqrt{9p^4}\cdot\sqrt{2}$
Simplify.	$40\cdot 3p^2\cdot\sqrt{3}$
	$120p^2\sqrt{3}$

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

	$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$
Multiply.	$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$
Simplify the radical.	$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$
Simplify.	$6\cdot 2y\sqrt[4]{35y}$
Multiply.	$12y\sqrt[4]{35y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$ ⓑ $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$.

Solution:

ⓐ $36x^3\sqrt{5}$ ⓑ $8y\sqrt[4]{3y^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$ ⓑ $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$.

Solution:

ⓐ $144y^2\sqrt{5y}$ ⓑ $-36\sqrt[4]{3a}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$ ⓑ $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$.

Solution:

ⓐ

	$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$
Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

ⓑ

	$\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$
Distribute.	$5\sqrt[3]{9}-\sqrt[3]{162}$
Simplify.	$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$
Simplify.	$5\sqrt[3]{9}-3\sqrt[3]{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{6}\left(1+3\sqrt{6}\right)$ ⓑ $\sqrt[3]{4}\left(-2-\sqrt[3]{6}\right)$.

Solution:

ⓐ $18+\sqrt{6}$ ⓑ $-2\sqrt[3]{4}-2\sqrt[3]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{8}\left(2-5\sqrt{8}\right)$ ⓑ $\sqrt[3]{3}\left(-\sqrt[3]{9}-\sqrt[3]{6}\right)$.

Solution:

ⓐ $-40+4\sqrt{2}$ ⓑ $-3-\sqrt[3]{18}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$.

Solution:

ⓐ

	$\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$
Multiply	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

ⓑ

	$\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$
Multiply.	$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$
Combine like terms.	$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6-3\sqrt{7}\right)\left(3+4\sqrt{7}\right)$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}-3\right)$.

Solution:

ⓐ $-66+15\sqrt{7}$
ⓑ $\sqrt[3]{x^2}-5\sqrt[3]{x}+6$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(2-3\sqrt{11}\right)\left(4-\sqrt{11}\right)$ ⓑ $\left(\sqrt[3]{x}+1\right)\left(\sqrt[3]{x}+3\right)$.

Solution:

ⓐ $41-14\sqrt{11}$
ⓑ $\sqrt[3]{x^2}+4\sqrt[3]{x}+3$

Example:
Exercise:

Problem: Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution:

$$\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$$

Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

Note:
Exercise:

Problem: Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$

Solution:

$1+9\sqrt{21}$

Note:
Exercise:

Problem: Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Solution:

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Note:
Special Products
Equation:

Binomial Squares	**Product of Conjugates**
$(a+b)^2=a^2+2ab+b^2$	$(a+b)(a-b)=a^2-b^2$
$(a-b)^2=a^2-2ab+b^2$	

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(2+\sqrt{3}\right)^2$ ⓑ $\left(4-2\sqrt{5}\right)^2$.

Solution:

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$(a+b)^2$ $\left(2+\sqrt{3}\right)^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2+2\ a\ b+b^2$ $2^2+2\cdot 2\cdot\sqrt{3}+\left(\sqrt{3}\right)^2$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$(a - b)^2$ $(4 - 2\sqrt{5})^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2 - 2\ a\ b + b^2$ $4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + (2\sqrt{5})^2$
Simplify.	$16 - 16\sqrt{5} + 4 \cdot 5$
	$16 - 16\sqrt{5} + 20$
Combine like terms.	$36 - 16\sqrt{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(10 + \sqrt{2}\right)^2$ ⓑ $\left(1 + 3\sqrt{6}\right)^2$.

Solution:

ⓐ $102 + 20\sqrt{2}$ ⓑ $55 + 6\sqrt{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6 - \sqrt{5}\right)^2$ ⓑ $\left(9 - 2\sqrt{10}\right)^2$.

Solution:

ⓐ $41 - 12\sqrt{5}$
ⓑ $121 - 36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example:
Exercise:

Problem: Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution:

	$\underset{(a-b)}{\left(5-2\sqrt{3}\right)}\underset{(a+b)}{\left(5+2\sqrt{3}\right)}$
Multiply, using the Product of Conjugates Pattern.	a^2-b^2 $5^2-\left(2\sqrt{3}\right)^2$
Simplify.	$25-4\cdot 3$
	13

Note:
Exercise:

Problem: Simplify: $\left(3-2\sqrt{5}\right)\left(3+2\sqrt{5}\right)$

Solution:

-11

Note:
Exercise:

Problem: Simplify: $\left(4+5\sqrt{7}\right)\left(4-5\sqrt{7}\right)$.

Solution:

-159

Note:
Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
    $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

Practice Makes Perfect

Add and Subtract Radical Expressions

In the following exercises, simplify.
Exercise:

Problem: ⓐ $8\sqrt{2} - 5\sqrt{2}$
ⓑ $5\sqrt[3]{m} + 2\sqrt[3]{m}$
ⓒ $8\sqrt[4]{m} - 2\sqrt[4]{n}$

Solution:

ⓐ $3\sqrt{2}$ ⓑ $7\sqrt[3]{m}$ ⓒ $6\sqrt[4]{m}$

Exercise:

Problem: ⓐ $7\sqrt{2} - 3\sqrt{2}$
ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

Exercise:

Problem: ⓐ $3\sqrt{5} + 6\sqrt{5}$
ⓑ $9\sqrt[3]{a} + 3\sqrt[3]{a}$
ⓒ $5\sqrt[4]{2z} + \sqrt[4]{2z}$

Solution:

ⓐ $9\sqrt{5}$ ⓑ $12\sqrt[3]{a}$ ⓒ $6\sqrt[4]{2z}$

Exercise:

Problem: ⓐ $4\sqrt{5} + 8\sqrt{5}$
ⓑ $\sqrt[3]{m} - 4\sqrt[3]{m}$
ⓒ $\sqrt{n} + 3\sqrt{n}$

Exercise:

Problem: ⓐ $3\sqrt{2a} - 4\sqrt{2a} + 5\sqrt{2a}$
ⓑ $5\sqrt[4]{3ab} - 3\sqrt[4]{3ab} - 2\sqrt[4]{3ab}$

Solution:

ⓐ $4\sqrt{2a}$ ⓑ 0

Exercise:

Problem: ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

Exercise:

Problem: ⓐ $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$
ⓑ $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$

Solution:

ⓐ $\sqrt{3c}$ ⓑ $\sqrt[3]{4pq}$

Exercise:

Problem: ⓐ $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$
ⓑ $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$

Exercise:

Problem: ⓐ $\sqrt{27}-\sqrt{75}$
ⓑ $\sqrt[3]{40}-\sqrt[3]{320}$
ⓒ $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$

Solution:

ⓐ $4\sqrt{3}$ ⓑ $-2\sqrt[3]{5}$ ⓒ $3\sqrt[3]{2}$

Exercise:

Problem: ⓐ $\sqrt{72}-\sqrt{98}$
ⓑ $\sqrt[3]{24}+\sqrt[3]{81}$
ⓒ $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

Exercise:

Problem: ⓐ $\sqrt{48}+\sqrt{27}$
ⓑ $\sqrt[3]{54}+\sqrt[3]{128}$
ⓒ $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$

Solution:

ⓐ $7\sqrt{3}$ ⓑ $7\sqrt[3]{2}$ ⓒ $3\sqrt[4]{5}$

Exercise:

Problem: ⓐ $\sqrt{45}+\sqrt{80}$
ⓑ $\sqrt[3]{81}-\sqrt[3]{192}$
ⓒ $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$

Exercise:

Problem: ⓐ $\sqrt{72a^5}-\sqrt{50a^5}$
ⓑ $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$

Solution:

ⓐ $a^2\sqrt{2a}$ ⓑ 0

Exercise:

Problem: ⓐ $\sqrt{48b^5}-\sqrt{75b^5}$
ⓑ $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

Exercise:

Problem: ⓐ $\sqrt{80c^7}-\sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$

Solution:

ⓐ $2c^3\sqrt{5c}$ ⓑ $14r^2\sqrt[4]{2r^2}$

Exercise:

Problem: ⓐ $\sqrt{96d^9}-\sqrt{24d^9}$
ⓑ $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$

Exercise:

Problem: $3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$

Solution:

$4y\sqrt{2}$

Exercise:

Problem: $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.
Exercise:

Problem: ⓐ $\left(-2\sqrt{3}\right)\left(3\sqrt{18}\right)$ ⓑ $\left(8\sqrt[3]{4}\right)\left(-4\sqrt[3]{18}\right)$

Solution:

ⓐ $-18\sqrt{6}$ ⓑ $-64\sqrt[3]{9}$

Exercise:

Problem: ⓐ $\left(-4\sqrt{5}\right)\left(5\sqrt{10}\right)$ ⓑ $\left(-2\sqrt[3]{9}\right)\left(7\sqrt[3]{9}\right)$

Exercise:

Problem: ⓐ $\left(5\sqrt{6}\right)\left(-\sqrt{12}\right)$ ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

Solution:

ⓐ $-30\sqrt{2})$ ⓑ $6\sqrt[4]{2}$

Exercise:

Problem: ⓐ $\left(-2\sqrt{7}\right)\left(-2\sqrt{14}\right)$ ⓑ $\left(-3\sqrt[4]{8}\right)\left(-5\sqrt[4]{6}\right)$

Exercise:

Problem: ⓐ $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$
ⓑ $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

Solution:

ⓐ $72z^2\sqrt{3}$ ⓑ $45x^2\sqrt[3]{2}$

Exercise:

Problem: ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Exercise:

Problem: ⓐ $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$
ⓑ $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

Solution:

ⓐ $-42z^5\sqrt{2z}$ ⓑ $-8y\sqrt[4]{6y}$

Exercise:

Problem: ⓐ $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$
ⓑ $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

Exercise:

Problem: ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$
ⓑ $\sqrt[3]{6}\left(4+\sqrt[3]{18}\right)$

Solution:

ⓐ $14+5\sqrt{7}$ ⓑ $4\sqrt[3]{6}+3\sqrt[3]{4}$

Exercise:

Problem: ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$
ⓑ $\sqrt[3]{3}\ \left(\sqrt[3]{9}+\sqrt[3]{18}\right)$

Exercise:

Problem: ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$
ⓑ $\sqrt[4]{3}\ \left(\sqrt[4]{54}+\sqrt[4]{18}\right)$

Solution:

ⓐ $44-3\sqrt{11}$ ⓑ $3\sqrt[4]{2}+\sqrt[4]{54}$

Exercise:

Problem: ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$
ⓑ $\sqrt[4]{2}\ \left(\sqrt[4]{12}+\sqrt[4]{24}\right)$

Exercise:

Problem: $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

Solution:

$60+2\sqrt{3}$

Exercise:

Problem: $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

Exercise:

Problem: ⓐ $\left(9-3\sqrt{2}\right)\left(6+4\sqrt{2}\right)$
ⓑ $\left(\sqrt[3]{x}-3\right)\left(\sqrt[3]{x}+1\right)$

Solution:

ⓐ $30+18\sqrt{2}$
ⓑ $\sqrt[3]{x^2}-2\sqrt[3]{x}-3$

Exercise:

Problem: ⓐ $\left(3-2\sqrt{7}\right)\left(5-4\sqrt{7}\right)$
ⓑ $\left(\sqrt[3]{x}-5\right)\left(\sqrt[3]{x}-3\right)$

Exercise:

Problem: ⓐ $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$
ⓑ $\left(2\sqrt[3]{x}+6\right)\left(\sqrt[3]{x}+1\right)$

Solution:

ⓐ $-54+13\sqrt{10}$
ⓑ $2\sqrt[3]{x^2}+8\sqrt[3]{x}+6$

Exercise:

Problem: ⓐ $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$
ⓑ $\left(3\sqrt[3]{x}+2\right)\left(\sqrt[3]{x}-2\right)$

Exercise:

Problem: $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

Solution:

$23+3\sqrt{30}$

Exercise:

Problem: $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

Exercise:

Problem: $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

Solution:

$-439-2\sqrt{77}$

Exercise:

Problem: $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

Exercise:

Problem: ⓐ $\left(3+\sqrt{5}\right)^2$ ⓑ $\left(2-5\sqrt{3}\right)^2$

Solution:

ⓐ $14+6\sqrt{5}$ ⓑ $79-20\sqrt{3}$

Exercise:

Problem: ⓐ $\left(4+\sqrt{11}\right)^2$ ⓑ $\left(3-2\sqrt{5}\right)^2$

Exercise:

Problem: ⓐ $\left(9-\sqrt{6}\right)^2$ ⓑ $\left(10+3\sqrt{7}\right)^2$

Solution:

ⓐ $87-18\sqrt{6}$
ⓑ $163+60\sqrt{7}$

Exercise:

Problem: ⓐ $\left(5-\sqrt{10}\right)^2$ ⓑ $\left(8+3\sqrt{2}\right)^2$

Exercise:

Problem: $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)$

Solution:

14

Exercise:

Problem: $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

Exercise:

Problem: $\left(4+9\sqrt{3}\right)\left(4-9\sqrt{3}\right)$

Solution:

-227

Exercise:

Problem: $\left(1+8\sqrt{2}\right)\left(1-8\sqrt{2}\right)$

Exercise:

Problem: $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

Solution:

19

Exercise:

Problem: $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

Exercise:

Problem: $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

Solution:

$\sqrt[3]{9x^2}-4$

Exercise:

Problem: $\left(\sqrt[3]{4x}+3\right)\left(\sqrt[3]{4x}-3\right)$

Mixed Practice

Exercise:

Problem: $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

Solution:

$5\sqrt{3}$

Exercise:

Problem: $\sqrt{175k^4}-\sqrt{63k^4}$

Exercise:

Problem: $\frac{5}{6}\sqrt{162} + \frac{3}{16}\sqrt{128}$

Solution:

$9\sqrt{2}$

Exercise:

Problem: $\sqrt[3]{24} + \sqrt[3]{/81}$

Exercise:

Problem: $\frac{1}{2}\ \sqrt[4]{80} - \frac{2}{3}\ \sqrt[4]{405}$

Solution:

$-\sqrt[4]{5}$

Exercise:

Problem: $8\sqrt[4]{13} - 4\sqrt[4]{13} - 3\sqrt[4]{13}$

Exercise:

Problem: $5\sqrt{12c^4} - 3\sqrt{27c^6}$

Solution:

$10c^2\sqrt{3} - 9c^3\sqrt{3}$

Exercise:

Problem: $\sqrt{80a^5} - \sqrt{45a^5}$

Exercise:

Problem: $\frac{3}{5}\sqrt{75} - \frac{1}{4}\sqrt{48}$

Solution:

$2\sqrt{3}$

Exercise:

Problem: $21\ \sqrt[3]{9} - 2\ \sqrt[3]{9}$

Exercise:

Problem: $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

Solution:

$17q^2$

Exercise:

Problem: $11\sqrt{11} - 10\sqrt{11}$

Exercise:

Problem: $\sqrt{3} \cdot \sqrt{21}$

Solution:

$3\sqrt{7}$

Exercise:

Problem: $\left(4\sqrt{6}\right)\left(-\sqrt{18}\right)$

Exercise:

Problem: $\left(7\sqrt[3]{4}\right)\left(-3\sqrt[3]{18}\right)$

Solution:

$-42\sqrt[3]{9}$

Exercise:

Problem: $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

Exercise:

Problem: $\left(\sqrt{29}\right)^2$

Solution:

29

Exercise:

Problem: $\left(-4\sqrt{17}\right)\left(-3\sqrt{17}\right)$

Exercise:

Problem: $\left(-4+\sqrt{17}\right)\left(-3+\sqrt{17}\right)$

Solution:

$29-7\sqrt{17}$

Exercise:

Problem: $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

Exercise:

Problem: $\left(6-3\sqrt{2}\right)^2$

Solution:

$72-36\sqrt{2}$

Exercise:

Problem: $\sqrt{3}\left(4-3\sqrt{3}\right)$

Exercise:

Problem: $\sqrt[3]{3}\left(2\sqrt[3]{9}+\sqrt[3]{18}\right)$

Solution:

$6+3\sqrt[3]{2}$

Exercise:

Problem: $\left(\sqrt{6}+\sqrt{3}\right)\left(\sqrt{6}+6\sqrt{3}\right)$

Writing Exercises

Exercise:

Problem: Explain the when a radical expression is in simplest form.

Solution:

Answers will vary.

Exercise:

Problem:

Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

Exercise:

Problem: ⓐ Explain why $\left(-\sqrt{n}\right)^2$ is always non-negative, for $n \geq 0$.
ⓑ Explain why $-\left(\sqrt{n}\right)^2$ is always non-positive, for $n \geq 0$.

Solution:

Answers will vary.

Exercise:

Problem: Use the binomial square pattern to simplify $\left(3+\sqrt{2}\right)^2$. Explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

like radicals
: Like radicals are radical expressions with the same index and the same radicand.

Divide Radical Expressions
By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review [link].
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review [link].
3. Multiply: $(7 + 3x)(7 - 3x)$.
   If you missed this problem, review [link].

Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Note:
Quotient Property of Radical Expressions
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
Equation:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{72x^3}}{\sqrt{162x}}$ ⓑ $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$.

Solution:

ⓐ

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.	$\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$
Simplify.	$\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$\frac{2x}{3}$

ⓑ

$$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$$

Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

Simplify the fraction under the radical.

$$\sqrt[3]{\frac{8}{x^3}}$$

Simplify the radical.

$$\frac{2}{x}$$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{50s^3}}{\sqrt{128s}}$ ⓑ $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

Solution:

ⓐ $\frac{5s}{8}$ ⓑ $\frac{2}{a}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{75q^5}}{\sqrt{108q}}$ ⓑ $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$.

Solution:

ⓐ $\frac{5q^2}{6}$ ⓑ $\frac{2}{b}$

Example:

Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ ⓑ $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$.

Solution:

ⓐ

	$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property.	$\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors in the fraction.	$\sqrt{\frac{49b^4}{a^2}}$
Simplify the radical.	$\frac{7b^2}{a}$

ⓑ

	$\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$
Rewrite using the quotient property.	$\sqrt[3]{\frac{-250mn^{-2}}{2m^{-2}n^4}}$
Simplify the fraction under the radical.	$\sqrt[3]{\frac{-125m^3}{n^6}}$
Simplify the radical.	$-\frac{5m}{n^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ ⓑ $\frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

Solution:

ⓐ $\frac{9x^2}{y^2}$ ⓑ $\frac{-4x}{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$ ⓑ $\dfrac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$.

Solution:

ⓐ $\dfrac{10n^3}{m}$ ⓑ $\dfrac{-3p}{q^2}$

Example:
Exercise:

Problem: Simplify: $\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution:

	$\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$\sqrt{\dfrac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9x^2y^2 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt{9x^2y^2} \cdot \sqrt{2x}$
Simplify.	$3xy\sqrt{2x}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

Solution:

$4xy\sqrt{2x}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Solution:

$4ab\sqrt{3b}$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

Note:

Rationalizing the Denominator
Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

Note:
Simplified Radical Expressions
A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $\left(\sqrt{a}\right)^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{4}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{20}}$ ⓒ $\frac{3}{\sqrt{6x}}$.

Solution:

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{3}{20}}$

The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$
Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

ⓒ

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	

	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{5}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{32}}$ ⓒ $\frac{2}{\sqrt{2x}}$.

Solution:

ⓐ $\frac{5\sqrt{3}}{3}$ ⓑ $\frac{\sqrt{6}}{8}$ ⓒ $\frac{\sqrt{2x}}{x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{6}{\sqrt{5}}$ ⓑ $\sqrt{\frac{7}{18}}$ ⓒ $\frac{5}{\sqrt{5x}}$.

Solution:

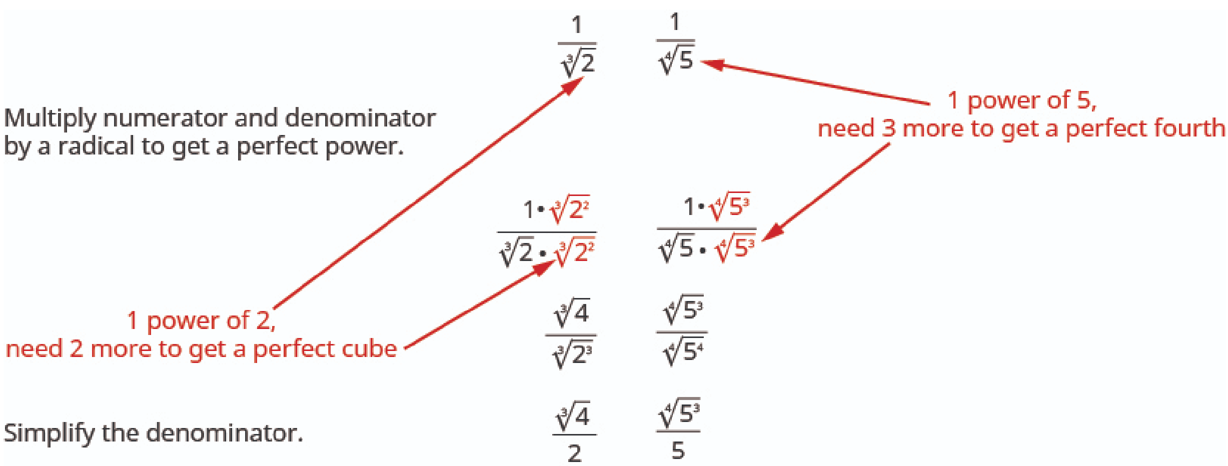
ⓐ $\frac{6\sqrt{5}}{5}$ ⓑ $\frac{\sqrt{14}}{6}$ ⓒ $\frac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the

radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,



We will use this technique in the next examples.

Example:
Exercise:

Problem: Simplify ⓐ $\frac{1}{\sqrt[3]{6}}$ ⓑ $\sqrt[3]{\frac{7}{24}}$ ⓒ $\frac{3}{\sqrt[3]{4x}}$.

Solution:

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[3]{\frac{7}{24}}$
The fraction is not a perfect cube, so	

rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7}\cdot\sqrt[3]{3^2}}{2\sqrt[3]{3}\cdot\sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2\cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

ⓒ

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	

	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of *x*.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{\sqrt[3]{7}}$ ⓑ $\sqrt[3]{\frac{5}{12}}$ ⓒ $\frac{5}{\sqrt[3]{9y}}$.

Solution:

ⓐ $\frac{\sqrt[3]{49}}{7}$ ⓑ $\frac{\sqrt[3]{90}}{6}$ ⓒ $\frac{5\sqrt[3]{3y^2}}{3y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{\sqrt[3]{2}}$ ⓑ $\sqrt[3]{\frac{3}{20}}$ ⓒ $\frac{2}{\sqrt[3]{25n}}$.

Solution:

ⓐ $\frac{\sqrt[3]{4}}{2}$ ⓑ $\frac{\sqrt[3]{150}}{10}$ ⓒ $\frac{2\sqrt[3]{5n^2}}{5n}$

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{\sqrt[4]{2}}$ ⓑ $\sqrt[4]{\frac{5}{64}}$ ⓒ $\frac{2}{\sqrt[4]{8x}}$.

Solution:

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$

Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[4]{\frac{5}{64}}$
The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{2^2}}{2\sqrt[4]{2^2}\cdot\sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{4}}{2\sqrt[4]{2^4}}$

Remember, $\sqrt[4]{2^4} = 2$.	$\frac{\sqrt[4]{20}}{2 \cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

ⓒ

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2 \cdot x^3}$. This will get us 4 factors of 2 and 4 factors of *x*.	$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2^4 x^4}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{\sqrt[4]{3}}$ ⓑ $\sqrt[4]{\frac{3}{64}}$ ⓒ $\frac{3}{\sqrt[4]{125x}}$.

Solution:

ⓐ $\frac{\sqrt[4]{27}}{3}$ ⓑ $\frac{\sqrt[4]{12}}{4}$ ⓒ $\frac{3\sqrt[4]{5x^3}}{5x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{\sqrt[4]{5}}$ ⓑ $\sqrt[4]{\frac{7}{128}}$ ⓒ $\frac{4}{\sqrt[4]{4x}}$

Solution:

ⓐ $\frac{\sqrt[4]{125}}{5}$ ⓑ $\frac{\sqrt[4]{224}}{8}$
ⓒ $\frac{\sqrt[4]{64x^3}}{x}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.
Equation:

$$(a-b)(a+b) \qquad\qquad \left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$$

$$a^2-b^2 \qquad\qquad 2^2-\left(\sqrt{5}\right)^2$$

$$4-5$$

$$-1$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

Example:
Exercise:

Problem: Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution:

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}$
Multiply the conjugates in the denominator.	$\frac{5\left(2+\sqrt{3}\right)}{2^2-\left(\sqrt{3}\right)^2}$

Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

Note:
Exercise:

Problem: Simplify: $\frac{3}{1-\sqrt{5}}$.

Solution:

$-\frac{3\left(1+\sqrt{5}\right)}{4}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{4-\sqrt{6}}$.

Solution:

$\frac{4+\sqrt{6}}{5}$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$
Simplify the denominator.	$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Solution:

$\frac{\sqrt{5}\left(\sqrt{x}-\sqrt{2}\right)}{x-2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Solution:

$\frac{\sqrt{10}\left(\sqrt{y}+\sqrt{3}\right)}{y-3}$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:

	$\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\dfrac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Solution:

$$\frac{\left(\sqrt{p}+\sqrt{2}\right)^2}{p-2}$$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

Solution:

$\frac{\left(\sqrt{q}-\sqrt{10}\right)^2}{q-10}$

Note:
Access these online resources for additional instruction and practice with dividing radical expressions.

- Rationalize the Denominator
- Dividing Radical Expressions and Rationalizing the Denominator
- Simplifying a Radical Expression with a Conjugate
- Rationalize the Denominator of a Radical Expression

Key Concepts

- **Quotient Property of Radical Expressions**
    - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
    $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **Simplified Radical Expressions**
    - A radical expression is considered simplified if there are:
        - no factors in the radicand that have perfect powers of the index

- no fractions in the radicand
- no radicals in the denominator of a fraction

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\frac{\sqrt{128}}{\sqrt{72}}$ ⓑ $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

Solution:

ⓐ $\frac{4}{3}$ ⓑ $\frac{4}{3}$

Exercise:

Problem: ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

Exercise:

Problem: ⓐ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ ⓑ $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$

Solution:

ⓐ $\frac{10m^2}{7}$ ⓑ $\frac{3}{y}$

Exercise:

Problem: ⓐ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ ⓑ $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{75r^3}}{\sqrt{108r^7}}$ ⓑ $\dfrac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$

Solution:

ⓐ $\dfrac{5}{6r^2}$ ⓑ $\dfrac{2x}{3}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{196q}}{\sqrt{484q^5}}$ ⓑ $\dfrac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ ⓑ $\dfrac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$

Solution:

ⓐ $\dfrac{6p}{q^2}$ ⓑ $-\dfrac{2a^2}{b}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ ⓑ $\dfrac{\sqrt[3]{-375y^4z^{-2}}}{\sqrt[3]{3y^{-2}z^4}}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ ⓑ $\dfrac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Solution:

ⓐ $\dfrac{8m^4}{3n^4}$ ⓑ $-\dfrac{x^2}{2y^2}$

Exercise:

Problem: ⓐ $\dfrac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd^{-1}}}$ ⓑ $\dfrac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$

Exercise:

Problem: $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$

Solution:

$4x^4\sqrt{7y}$

Exercise:

Problem: $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

Exercise:

Problem: $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$

Solution:

$2ab\sqrt[3]{2a}$

Exercise:

Problem: $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

Exercise:

Problem: ⓐ $\frac{10}{\sqrt{6}}$ ⓑ $\sqrt{\frac{4}{27}}$ ⓒ $\frac{10}{\sqrt{5x}}$

Solution:

ⓐ $\frac{5\sqrt{6}}{3}$ ⓑ $\frac{2\sqrt{3}}{9}$ ⓒ $\frac{2\sqrt{5x}}{x}$

Exercise:

Problem: ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

Exercise:

Problem: ⓐ $\frac{6}{\sqrt{7}}$ ⓑ $\sqrt{\frac{8}{45}}$ ⓒ $\frac{12}{\sqrt{3p}}$

Solution:

ⓐ $\frac{6\sqrt{7}}{7}$ ⓑ $\frac{2\sqrt{10}}{15}$ ⓒ $\frac{4\sqrt{3p}}{p}$

Exercise:

Problem: ⓐ $\frac{4}{\sqrt{5}}$ ⓑ $\sqrt{\frac{27}{80}}$ ⓒ $\frac{18}{\sqrt{6q}}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[3]{5}}$ ⓑ $\sqrt[3]{\frac{5}{24}}$ ⓒ $\frac{4}{\sqrt[3]{36a}}$

Solution:

ⓐ $\frac{\sqrt[3]{25}}{5}$ ⓑ $\frac{\sqrt[3]{45}}{6}$ ⓒ $\frac{2\sqrt[3]{6a^2}}{3a}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[3]{3}}$ ⓑ $\sqrt[3]{\frac{5}{32}}$ ⓒ $\frac{7}{\sqrt[3]{49b}}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

Solution:

ⓐ $\frac{\sqrt[3]{121}}{11}$ ⓑ $\frac{\sqrt[3]{28}}{6}$ ⓒ $\frac{\sqrt[3]{9x}}{x}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[3]{13}}$ ⓑ $\sqrt[3]{\frac{3}{128}}$ ⓒ $\frac{3}{\sqrt[3]{6y^2}}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[4]{7}}$ ⓑ $\sqrt[4]{\frac{5}{32}}$ ⓒ $\frac{4}{\sqrt[4]{4x^2}}$

Solution:

ⓐ $\frac{\sqrt[4]{343}}{7}$ ⓑ $\frac{\sqrt[4]{40}}{4}$ ⓒ $\frac{2\sqrt[4]{4x^2}}{x}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[4]{9}}$ ⓑ $\sqrt[4]{\frac{25}{128}}$ ⓒ $\frac{6}{\sqrt[4]{27a}}$

Solution:

ⓐ $\frac{\sqrt[4]{9}}{3}$ ⓑ $\frac{\sqrt[4]{50}}{4}$ ⓒ $\frac{2\sqrt[4]{3a^2}}{a}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[4]{8}}$ ⓑ $\sqrt[4]{\frac{27}{128}}$ ⓒ $\frac{16}{\sqrt[4]{64b^2}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

Exercise:

Problem: $\frac{8}{1-\sqrt{5}}$

Solution:

$$-2\left(1+\sqrt{5}\right)$$

Exercise:

Problem: $\frac{7}{2-\sqrt{6}}$

Exercise:

Problem: $\frac{6}{3-\sqrt{7}}$

Solution:

$$3\left(3+\sqrt{7}\right)$$

Exercise:

Problem: $\frac{5}{4-\sqrt{11}}$

Exercise:

Problem: $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

Solution:

$$\frac{\sqrt{3}\left(\sqrt{m}+\sqrt{5}\right)}{m-5}$$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

Solution:

$$\frac{\sqrt{2}\left(\sqrt{x}+\sqrt{6}\right)}{x-6}$$

Exercise:

Problem: $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

Exercise:

Problem: $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

Solution:

$$\frac{\left(\sqrt{r}+\sqrt{5}\right)^2}{r-5}$$

Exercise:

Problem: $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

Exercise:

Problem: $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Solution:

$$\frac{\left(\sqrt{x}+2\sqrt{2}\right)^2}{x-8}$$

Exercise:

Problem: $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Writing Exercises

Exercise:

Problem:

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.
ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.
ⓒ Why are the two methods of simplifying square roots different?

Solution:

Answers will vary.

Exercise:

Problem:

Explain what is meant by the word rationalize in the phrase, “rationalize a denominator.”

Exercise:

Problem:

Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an expression with no radicals.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

rationalizing the denominator
: Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Solve Radical Equations
By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review [link].
2. Solve: $2x-5=0$.
   If you missed this problem, review [link].
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review [link].

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Note:
Radical Equation
An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the n^{th} power. This will eliminate the radical.

Equation:

$$\text{For } a \geq 0,\ \left(\sqrt[n]{a}\right)^n = a.$$

Example:
How to Solve a Radical Equation
Exercise:

Problem: Solve: $\sqrt{5n-4} - 9 = 0$.

Solution:

Step 1. Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4} - 9 = 0$ $\sqrt{5n-4} - 9 + 9 = 0 + 9$ $\sqrt{5n-4} = 9$
Step 2. Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$\left(\sqrt{5n-4}\right)^2 = (9)^2$
Step 3. Solve the new equation.	Remember, $\left(\sqrt{a}\right)^2 = a$.	$5n - 4 = 81$ $5n = 85$ $n = 17$
Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4} - 9 = 0$ $\sqrt{5(17)-4} - 9 \stackrel{?}{=} 0$ $\sqrt{85-4} - 9 \stackrel{?}{=} 0$ $\sqrt{81} - 9 \stackrel{?}{=} 0$ $9 - 9 \stackrel{?}{=} 0$ $0 = 0$ ✓ The solution is $n = 17$.

Note:
Exercise:

Problem: Solve: $\sqrt{3m+2}-5=0.$

Solution:

$m=\frac{23}{3}$

Note:
Exercise:

Problem: Solve: $\sqrt{10z+1}-2=0.$

Solution:

$z=\frac{3}{10}$

Note:
Solve a radical equation with one radical.

Isolate the radical on one side of the equation.
Raise both sides of the equation to the power of the index.
Solve the new equation.
Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example:
Exercise:

Problem: Solve: $\sqrt{9k-2}+1=0.$

Solution:

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Note:
Exercise:

Problem: Solve: $\sqrt{2r-3}+5=0$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{7s-3}+2=0$.

Solution:

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Note:
Binomial Squares

Equation:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Don’t forget the middle term!

Example:
Exercise:

Problem: Solve: $\sqrt{p-1}+1=p$.

Solution:

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$

Factor the right side.	$0 = (p - 1)(p - 2)$
Use the Zero Product Property.	$0 = p - 1 \quad 0 = p - 2$
Solve each equation.	$p = 1 \quad p = 2$
Check the answers.	
$p = 1$: $\sqrt{p-1}+1=p$; $\sqrt{1-1}+1 \stackrel{?}{=} 1$; $\sqrt{0}+1 \stackrel{?}{=} 1$; $1 = 1 ✓$ $p = 2$: $\sqrt{p-1}+1=p$; $\sqrt{2-1}+1 \stackrel{?}{=} 2$; $\sqrt{1}+1 \stackrel{?}{=} 2$; $2 = 2 ✓$	
	The solutions are $p = 1, \; p = 2.$

Note:
Exercise:

Problem: Solve: $\sqrt{x-2}+2=x$.

Solution:

$x = 2, x = 3$

Note:
Exercise:

Problem: Solve: $\sqrt{y-5}+5=y$.

Solution:

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.
Equation:

$$\left(\sqrt[3]{a}\right)^3 = a$$

Example:
Exercise:

Problem: Solve: $\sqrt[3]{5x+1} + 8 = 4$.

Solution:

	$\sqrt[3]{5x+1} + 8 = 4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1} = -4$
Cube both sides of the equation.	$\left(\sqrt[3]{5x+1}\right)^3 = (-4)^3$
Simplify.	$5x + 1 = -64$
Solve the equation.	$5x = -65$
	$x = -13$
Check the answer.	

$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Note:
Exercise:

Problem: Solve: $\sqrt[3]{4x-3}+8=5$

Solution:

$x=-6$

Note:
Exercise:

Problem: Solve: $\sqrt[3]{6x-10}+1=-3$

Solution:

$x=-9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n=a^{m\cdot n}$, we have for example,
Equation:

$$\left(x^{\frac{1}{2}}\right)^2=x,\quad \left(x^{\frac{1}{3}}\right)^3=x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example:
Exercise:

Problem: Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x-2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x = 6 \quad (3x-2)^{\frac{1}{4}} + 3 = 5$ $(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $2 + 3 \stackrel{?}{=} 5$ $5 = 5 \checkmark$	
	The solution is $x = 6$.

Note:
Exercise:

Problem: Solve: $(9x+9)^{\frac{1}{4}}-2=1$.

Solution:

$x=8$

Note:
Exercise:

Problem: Solve: $(4x-8)^{\frac{1}{4}}+5=7$.

Solution:

$x=6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

Example:
Exercise:

Problem: Solve: $\sqrt{r+4}-r+2=0$.

Solution:

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	

	$\left(\sqrt{r+4}\right)^2 = (r-2)^2$
Simplify and then solve the equation	$r+4 = r^2 - 4r + 4$
It is a quadratic equation, so get zero on one side.	$0 = r^2 - 5r$
Factor the right side.	$0 = r\left(r-5\right)$
Use the Zero Product Property.	$0 = r \quad 0 = r-5$
Solve the equation.	$r = 0 \quad r = 5$
Check your answer.	
$r=0,\quad \sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2 \stackrel{?}{=} 0$ $\sqrt{4}+2 \stackrel{?}{=} 0$ $4 \neq 0$ $r=5,\quad \sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2 \stackrel{?}{=} 0$ $\sqrt{9}-3 \stackrel{?}{=} 0$ $0=0\checkmark$	The solution is $r = 5$.
	$r = 0$ is an extraneous solution.

Note:
Exercise:

Problem: Solve: $\sqrt{m+9} - m + 3 = 0$.

Solution:

$m = 7$

Note:
Exercise:

Problem: Solve: $\sqrt{n+1} - n + 1 = 0$.

Solution:

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example:
Exercise:

Problem: Solve: $3\sqrt{3x-5}-8=4$.

Solution:

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$\left(\sqrt{3x-5}\right)^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	

$x = 7$ $\quad 3\sqrt{3x-5} - 8 = 4$ $3\sqrt{3(7)-5} - 8 \stackrel{?}{=} 4$ $3\sqrt{21-5} - 8 \stackrel{?}{=} 4$ $3\sqrt{16} - 8 \stackrel{?}{=} 4$ $3(4) - 8 \stackrel{?}{=} 4$ $4 = 4\,\checkmark$	
	The solution is $x = 7$.

Note:
Exercise:

Problem: Solve: $2\sqrt{4a+4} - 16 = 16$.

Solution:

$a = 63$

Note:
Exercise:

Problem: Solve: $3\sqrt{2b+3} - 25 = 50$.

Solution:

$b = 311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example:
Exercise:

Problem: Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.
Since the index is 3, cube both sides of the equation.
Simplify, then solve the new equation.

$$\begin{aligned} \sqrt[3]{4x-3} &= \sqrt[3]{3x+2} \\ \left(\sqrt[3]{4x-3}\right)^3 &= \left(\sqrt[3]{3x+2}\right)^3 \\ 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \\ \text{The solution is } x &= 5. \end{aligned}$$

Check the answer.
We leave it to you to show that 5 checks!

Note:
Exercise:

Problem: Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

Solution:

$x=3$

Note:
Exercise:

Problem: Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Solution:

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

Example:
How to Solve a Radical Equation
Exercise:

Problem: Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution:

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides. Simplify—be very careful as you multiply!	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Step 3. Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1=m+9$ $2\sqrt{m}=8$ $\sqrt{m}=4$ $\left(\sqrt{m}\right)^2=(4)^2$ $m=16$
Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5\checkmark$ The solution is $m=16$.

Note:
Exercise:

Problem: Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Solution:

$x=4$

Note:
Exercise:

Problem: Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Solution:

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Note:
Solve a radical equation.

Isolate one of the radical terms on one side of the equation.
Raise both sides of the equation to the power of the index.

Are there any more radicals?	If yes, repeat Step 1 and Step 2 again.	If no, solve the new equation.

Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example:
Exercise:

Problem: Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(\overset{a-b}{3q-6}\right)^2$ $6^2\left(\sqrt{q-2}\right)^2=\overset{a^2}{(3q)^2}\overset{-}{-}\overset{2ab}{2\cdot 3q\cdot 6}\overset{+b^2}{+6^2}$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$

Factor the right side.	$0 = 9(q^2 - 8q + 12)$ $0 = 9(q-6)(q-2)$
Use the Zero Product Property.	$q - 6 = 0 \quad q - 2 = 0$ $q = 6 \quad q = 2$
The checks are left to you.	The solutions are $q = 6$ and $q = 2$.

Note:
Exercise:

Problem: Solve: $\sqrt{x-1} + 2 = \sqrt{2x+6}$

Solution:

$x = 5$

Note:
Exercise:

Problem: Solve: $\sqrt{x} + 2 = \sqrt{3x+4}$

Solution:

$x = 0 \; x = 4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Note:
Use a problem solving strategy for applications with formulas.

Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
Identify what we are looking for.
Name what we are looking for by choosing a variable to represent it.
Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Note:
Falling Objects
On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula
Equation:

$$t=\frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

	$t=\frac{\sqrt{h}}{4}$

	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example:
Exercise:

Problem:

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let $t =$ time.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5. Solve the equation.	

	$t = \frac{20}{4}$
	$t = 5$
Step 6. Check the answer in the problem and make sure it makes sense.	$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

Note:
Exercise:

Problem:

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Solution:

9 seconds

Note:
Exercise:

Problem:

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Solution:

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Note:
Skid Marks and Speed of a Car
If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula
Equation:

$$s = \sqrt{24d}$$

Example:
Exercise:

Problem:

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Step 1. Read the problem	

Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let s = the speed.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5. Solve the equation.	$s = \sqrt{4,560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277... ✓$
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Note:
Exercise:

Problem:

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

42.7 feet

Note:
Exercise:

Problem:

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

54.1 feet

Note:
Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2=a^2+2ab+b^2$
  $(a-b)^2=a^2-2ab+b^2$
- **Solve a Radical Equation**

  Isolate one of the radical terms on one side of the equation.
  Raise both sides of the equation to the power of the index.

Are there any more radicals?	If yes, repeat Step 1 and Step 2 again.	If no, solve the new equation.

Check the answer in the original equation.

- **Problem Solving Strategy for Applications with Formulas**

  Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Identify what we are looking for.
  Name what we are looking for by choosing a variable to represent it.
  Translate into an equation by writing the appropriate formula or model for the situation.
  Substitute in the given information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

- **Falling Objects**
  - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.

- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.
Exercise:

Problem: $\sqrt{5x - 6} = 8$

Solution:

$m = 14$

Exercise:

Problem: $\sqrt{4x - 3} = 7$

Exercise:

Problem: $\sqrt{5x + 1} = -3$

Solution:

no solution

Exercise:

Problem: $\sqrt{3y-4}=-2$

Exercise:

Problem: $\sqrt[3]{2x}=-2$

Solution:

$x=-4$

Exercise:

Problem: $\sqrt[3]{4x-1}=3$

Exercise:

Problem: $\sqrt{2m-3}-5=0$

Solution:

$m=14$

Exercise:

Problem: $\sqrt{2n-1}-3=0$

Exercise:

Problem: $\sqrt{6v-2}-10=0$

Solution:

$v=17$

Exercise:

Problem: $\sqrt{12u+1}-11=0$

Exercise:

Problem: $\sqrt{4m+2}+2=6$

Solution:

$m=\frac{7}{2}$

Exercise:

Problem: $\sqrt{6n+1}+4=8$

Exercise:

Problem: $\sqrt{2u-3}+2=0$

Solution:

no solution

Exercise:

Problem: $\sqrt{5v-2}+5=0$

Exercise:

Problem: $\sqrt{u-3}-3=u$

Solution:

$u=3, u=4$

Exercise:

Problem: $\sqrt{v-10}+10=v$

Exercise:

Problem: $\sqrt{r-1}=r-1$

Solution:

$r=1, r=2$

Exercise:

Problem: $\sqrt{s-8}=s-8$

Exercise:

Problem: $\sqrt[3]{6x+4}=4$

Solution:

$x=10$

Exercise:

Problem: $\sqrt[3]{11x+4}=5$

Exercise:

Problem: $\sqrt[3]{4x+5}-2=-5$

Solution:

$x=-8$

Exercise:

Problem: $\sqrt[3]{9x-1}-1=-5$

Exercise:

Problem: $(6x+1)^{\frac{1}{2}}-3=4$

Solution:

$x=8$

Exercise:

Problem: $(3x-2)^{\frac{1}{2}}+1=6$

Exercise:

Problem: $(8x+5)^{\frac{1}{3}}+2=-1$

Solution:

$x=-4$

Exercise:

Problem: $(12x-5)^{\frac{1}{3}}+8=3$

Exercise:

Problem: $(12x-3)^{\frac{1}{4}}-5=-2$

Solution:

$x=7$

Exercise:

Problem: $(5x-4)^{\frac{1}{4}}+7=9$

Exercise:

Problem: $\sqrt{x+1}-x+1=0$

Solution:

$x=3$

Exercise:

Problem: $\sqrt{y+4}-y+2=0$

Exercise:

Problem: $\sqrt{z+100}-z=-10$

Solution:

$z=21$

Exercise:

Problem: $\sqrt{w+25}-w=-5$

Exercise:

Problem: $3\sqrt{2x-3}-20=7$

Solution:

$x=42$

Exercise:

Problem: $2\sqrt{5x+1}-8=0$

Exercise:

Problem: $2\sqrt{8r+1}-8=2$

Solution:

$r=3$

Exercise:

Problem: $3\sqrt{7y+1}-10=8$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

Exercise:

Problem: $\sqrt{3u+7}=\sqrt{5u+1}$

Solution:

$u=3$

Exercise:

Problem: $\sqrt{4v+1}=\sqrt{3v+3}$

Exercise:

Problem: $\sqrt{8+2r}=\sqrt{3r+10}$

Solution:

$r=-2$

Exercise:

Problem: $\sqrt{10+2c}=\sqrt{4c+16}$

Exercise:

Problem: $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

Solution:

$x = 1$

Exercise:

Problem: $\sqrt[3]{8x-5} = \sqrt[3]{3x+5}$

Exercise:

Problem: $\sqrt[3]{2x^2+9x-18} = \sqrt[3]{x^2+3x-2}$

Solution:

$x = -8, x = 2$

Exercise:

Problem: $\sqrt[3]{x^2-x+18} = \sqrt[3]{2x^2-3x-6}$

Exercise:

Problem: $\sqrt{a} + 2 = \sqrt{a+4}$

Solution:

$a = 0$

Exercise:

Problem: $\sqrt{r} + 6 = \sqrt{r+8}$

Exercise:

Problem: $\sqrt{u} + 1 = \sqrt{u+4}$

Solution:

$u = \frac{9}{4}$

Exercise:

Problem: $\sqrt{x} + 1 = \sqrt{x+2}$

Exercise:

Problem: $\sqrt{a+5}-\sqrt{a}=1$

Solution:

$a=4$

Exercise:

Problem: $-2=\sqrt{d-20}-\sqrt{d}$

Exercise:

Problem: $\sqrt{2x+1}=1+\sqrt{x}$

Solution:

$x=0\ x=4$

Exercise:

Problem: $\sqrt{3x+1}=1+\sqrt{2x-1}$

Exercise:

Problem: $\sqrt{2x-1}-\sqrt{x-1}=1$

Solution:

$x=1\ x=5$

Exercise:

Problem: $\sqrt{x+1}-\sqrt{x-2}=1$

Exercise:

Problem: $\sqrt{x+7}-\sqrt{x-5}=2$

Solution:

$x=9$

Exercise:

Problem: $\sqrt{x+5}-\sqrt{x-3}=2$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Solution:

8.7 feet

Exercise:

Problem:

Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

Exercise:

Problem:

Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

Solution:

4.7 seconds

Exercise:

Problem:

Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

Exercise:

Problem:

Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

72 feet

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

Exercise:

Problem: Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

Solution:

Answers will vary.

Exercise:

Problem:

ⓐ Solve the equation $\sqrt{r+4}-r+2=0$.
ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

radical equation
: An equation in which a variable is in the radicand of a radical expression is called a radical equation.

Use Radicals in Functions
By the end of this section, you will be able to:

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Note:
Before you get started, take this readiness quiz.

1. Solve: $1-2x\geq 0$.
   If you missed this problem, review [link].
2. For $f\left(x\right)=3x-4$, evaluate $f\left(2\right)$, $f\left(-1\right)$, $f\left(0\right)$.
   If you missed this problem, review [link].
3. Graph $f\left(x\right)=\sqrt{x}$. State the domain and range of the function in interval notation.
   If you missed this problem, review [link].

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

The square root function is $f\left(x\right)=\sqrt{x}$.

The cube root function is $f\left(x\right)=\sqrt[3]{x}$.

Note:
Radical Function
A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of *f*(*x*) for a given value of *x* just as we did in our previous work with functions.

Example:
Exercise:

Problem: For the function $f(x) = \sqrt{2x-1}$, find ⓐ $f(5)$ ⓑ $f(-2)$.

Solution:

ⓐ

	$f(x) = \sqrt{2x-1}$
To evaluate $f(5)$, substitute 5 for x.	$f(5) = \sqrt{2 \cdot 5 - 1}$
Simplify.	$f(5) = \sqrt{9}$
Take the square root.	$f(5) = 3$

ⓑ

	$f(x) = \sqrt{2x-1}$
To evaluate $f(-2)$, substitute -2 for x.	$f(-2) = \sqrt{2(-2)-1}$
Simplify.	$f(-2) = \sqrt{-5}$

Since the square root of a negative number is not a real number, the function does not have a value at $x = -2$.

Note:
Exercise:

Problem: For the function $f(x) = \sqrt{3x-2}$, find ⓐ $f(6)$ ⓑ $f(0)$.

Solution:

ⓐ $f(6) = 4$ ⓑ no value at $x = 0$

Note:
Exercise:

Problem: For the function $g(x) = \sqrt{5x+5}$, find ⓐ $g(4)$ ⓑ $g(-3)$.

Solution:

ⓐ $g(4) = 5$ ⓑ no value at $f(-3)$

We follow the same procedure to evaluate cube roots.

Example:
Exercise:

Problem: For the function $g(x) = \sqrt[3]{x-6}$, find ⓐ $g(14)$ ⓑ $g(-2)$.

Solution:

ⓐ

	$g(x) = \sqrt[3]{x-6}$
To evaluate $g(14)$, substitute 14 for x.	$g(14) = \sqrt[3]{14-6}$
Simplify.	$g(14) = \sqrt[3]{8}$
Take the cube root.	$g(14) = 2$

ⓑ

	$g(x) = \sqrt[3]{x-6}$
To evaluate $g(-2)$, substitute -2 for x.	$g(-2) = \sqrt[3]{-2-6}$
Simplify.	$g(-2) = \sqrt[3]{-8}$
Take the cube root.	$g(-2) = -2$

Note:
Exercise:

Problem: For the function $g(x) = \sqrt[3]{3x-4}$, find ⓐ $g(4)$ ⓑ $g(1)$.

Solution:

ⓐ $g(4)=2$ ⓑ $g(1)=-1$

Note:
Exercise:

Problem: For the function $h\left(x\right)=\sqrt[3]{5x-2}$, find ⓐ $h\left(2\right)$ ⓑ $h\left(-5\right)$.

Solution:

ⓐ $h(2)=2$
ⓑ $h(-5)=-3$

The next example has fourth roots.

Example:
Exercise:

Problem: For the function $f\left(x\right)=\sqrt[4]{5x-4}$, find ⓐ $f\left(4\right)$ ⓑ $f\left(-12\right)$

Solution:

ⓐ

	$f\left(x\right)=\sqrt[4]{5x-4}$
To evaluate $f\left(4\right)$, substitute 4 for x.	$f\left(4\right)=\sqrt[4]{5\cdot 4-4}$
Simplify.	$f\left(4\right)=\sqrt[4]{16}$
Take the fourth root.	$f\left(4\right)=2$

ⓑ

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x. $\quad f(-12) = \sqrt[4]{5(-12)-4}$

Simplify. $\quad f(-12) = \sqrt[4]{-64}$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

Note:
Exercise:

Problem: For the function $f(x) = \sqrt[4]{3x+4}$, find ⓐ $f(4)$ ⓑ $f(-1)$.

Solution:

ⓐ $f(4) = 2$ ⓑ $f(-1) = 1$

Note:
Exercise:

Problem: For the function $g(x) = \sqrt[4]{5x+1}$, find ⓐ $g(16)$ ⓑ $g(3)$.

Solution:

ⓐ $g(16) = 3$ ⓑ $g(3) = 2$

Find the Domain of a Radical Function

To find the domain and range of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Note:
Properties of $\sqrt[n]{a}$
When *n* is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Note:
Domain of a Radical Function
When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
When the **index** of the radical is **odd**, the radicand can be any real number.

Example:
Exercise:

Problem:

Find the domain of the function, $f\left(x\right)=\sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f\left(x\right)=\sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \tfrac{4}{3} \end{aligned}$$

The domain of $f\left(x\right)=\sqrt{3x-4}$ is all values $x\ge \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3},\infty \right)$.

Note:
Exercise:

Problem:

Find the domain of the function, $f\left(x\right)=\sqrt{6x-5}$. Write the domain in interval notation.

Solution:

$\left[\frac{5}{6},\infty \right)$

Note:
Exercise:

Problem:

Find the domain of the function, $f\left(x\right)=\sqrt{4-5x}$. Write the domain in interval notation.

Solution:

$\left(-\infty ,\frac{4}{5}\right]$

Example:
Exercise:

Problem:

Find the domain of the function, $g\left(x\right)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Since the function, $g(x) = \sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x - 1 > 0$ and solve.

$$\begin{array}{lrcl} & x-1 & > & 0 \\ \text{Solve.} & x & > & 1 \end{array}$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x - 1 = 0$ to find the value that must be eliminated from the domain.

$$\begin{array}{lrcl} & x-1 & = & 0 \\ \text{Solve.} & x & = & 1 \text{ so } x \neq 1 \text{ in the domain.} \end{array}$$

Putting this together we get the domain is $x > 1$ and we write it as $(1, \infty)$.

Note:
Exercise:

Problem:

Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Solution:

$(-3, \infty)$

Note:
Exercise:

Problem:

Find the domain of the function, $h\left(x\right)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Solution:

$\left(5,\infty\right)$

The next example involves a cube root and so will require different thinking.

Example:
Exercise:

Problem:

Find the domain of the function, $f\left(x\right)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f\left(x\right)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $\left(-\infty,\infty\right)$.

The domain of $f\left(x\right)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $\left(-\infty,\infty\right)$.

Note:
Exercise:

Problem:

Find the domain of the function, $f\left(x\right)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Solution:

$\left(-\infty,\infty\right)$

Note:
Exercise:

Problem:

Find the domain of the function, $g\left(x\right)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Solution:

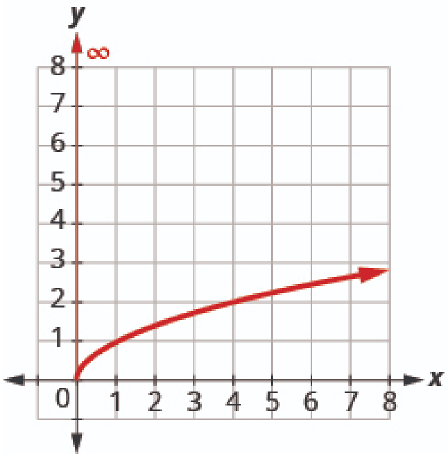
$\left(-\infty,\infty\right)$

Graph Radical Functions

Before we graph any radical function, we first find the domain of the function. For the function, $f\left(x\right)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\ge 0$ and we write this in interval notation as $\left[0,\infty\right)$.

Previously we used point plotting to graph the function, $f\left(x\right)=\sqrt{x}$. We chose *x*-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Once we see the graph, we can find the range of the function. The *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Example:
Exercise:

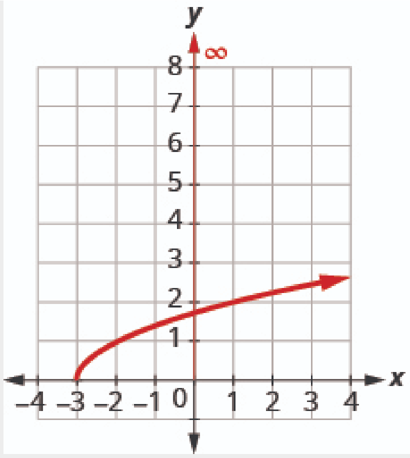
Problem: For the function $f(x) = \sqrt{x+3}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x + 3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.

ⓑ To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x) = \sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

ⓒ Looking at the graph, we see the *y*-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

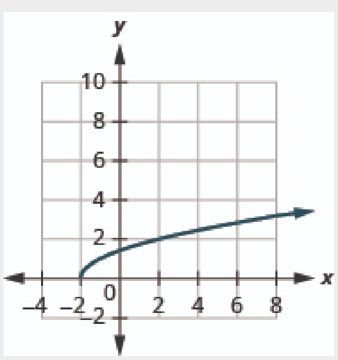
Note:
Exercise:

Problem:

For the function $f(x) = \sqrt{x+2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ domain: $[-2, \infty)$
ⓑ



ⓒ range: $[0, \infty)$

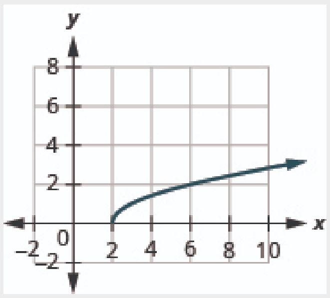
Note:
Exercise:

Problem:

For the function $f\left(x\right)=\sqrt{x-2}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ domain: $\left[2,\infty \right)$
ⓑ



ⓒ range: $\left[0,\infty \right)$

In our previous work graphing functions, we graphed $f\left(x\right)={x}^{3}$ but we did not graph the function $f\left(x\right)=\sqrt[3]{x}$. We will do this now in the next example.

Example:
Exercise:

Problem:

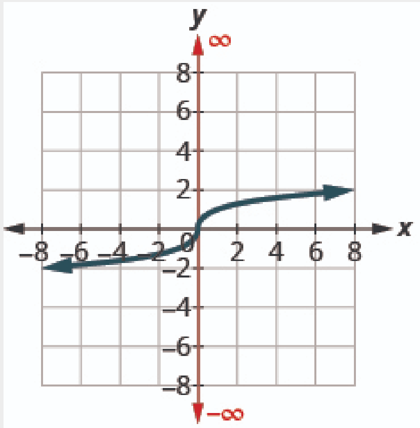
For the function $f\left(x\right)=\sqrt[3]{x}$, ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as

$(-\infty, \infty)$

ⓑ To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x) = \sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

ⓒ Looking at the graph, we see the *y*-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

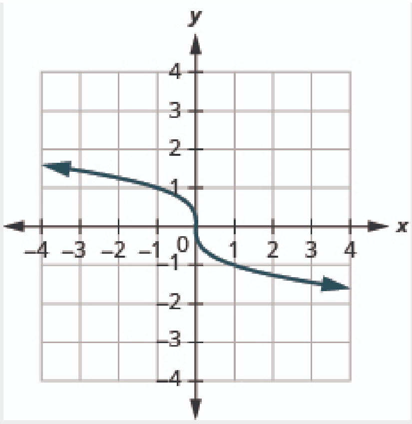
Note:
Exercise:

Problem: For the function $f\left(x\right) = -\sqrt[3]{x}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ range: $(-\infty, \infty)$

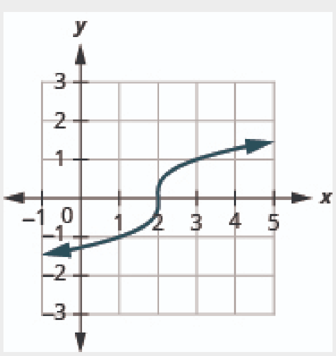
Note:
Exercise:

Problem: For the function $f(x) = \sqrt[3]{x-2}$,

ⓐ find the domain ⓑ graph the function ⓒ use the graph to determine the range.

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ range: $(-\infty, \infty)$

Note:
Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When *n* is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Practice Makes Perfect

Evaluate a Radical Function

In the following exercises, evaluate each function.
Exercise:

Problem: $f\left(x\right)=\sqrt{4x-4}$, find
ⓐ $f\left(5\right)$
ⓑ $f\left(0\right)$.

Solution:

ⓐ $f(5)=4$ ⓑ no value at $x=0$

Exercise:

Problem: $f\left(x\right)=\sqrt{6x-5}$, find
ⓐ $f\left(5\right)$
ⓑ $f\left(-1\right)$.

Exercise:

Problem: $g\left(x\right)=\sqrt{6x+1}$, find
ⓐ $g\left(4\right)$
ⓑ $g\left(8\right)$.

Solution:

ⓐ $g(4)=5$ ⓑ $g(8)=7$

Exercise:

Problem: $g\left(x\right)=\sqrt{3x+1}$, find
ⓐ $g\left(8\right)$
ⓑ $g\left(5\right)$.

Exercise:

Problem: $F\left(x\right)=\sqrt{3-2x}$, find
ⓐ $F\left(1\right)$
ⓑ $F\left(-11\right)$.

Solution:

ⓐ $F(1)=1$ ⓑ $F(-11)=5$

Exercise:

Problem: $F\left(x\right)=\sqrt{8-4x}$, find
ⓐ $F\left(1\right)$
ⓑ $F\left(-2\right)$.

Exercise:

Problem: $G(x) = \sqrt{5x-1}$, find
ⓐ $G(5)$
ⓑ $G(2)$.

Solution:

ⓐ $G(5) = 2\sqrt{6}$ ⓑ $G(2) = 3$

Exercise:

Problem: $G(x) = \sqrt{4x+1}$, find
ⓐ $G(11)$
ⓑ $G(2)$.

Exercise:

Problem: $g(x) = \sqrt[3]{2x-4}$, find
ⓐ $g(6)$
ⓑ $g(-2)$.

Solution:

ⓐ $g(6) = 2$ ⓑ $g(-2) = -2$

Exercise:

Problem: $g(x) = \sqrt[3]{7x-1}$, find
ⓐ $g(4)$
ⓑ $g(-1)$.

Exercise:

Problem: $h(x) = \sqrt[3]{x^2-4}$, find
ⓐ $h(-2)$
ⓑ $h(6)$.

Solution:

ⓐ $h(-2) = 0$ ⓑ $h(6) = 2\sqrt[3]{4}$

Exercise:

Problem: $h\left(x\right)=\sqrt[3]{x^2+4}$, find
ⓐ $h\left(-2\right)$
ⓑ $h(6)$.

Exercise:

Problem: For the function
$f\left(x\right)=\sqrt[4]{2x^3}$, find
ⓐ $f\left(0\right)$
ⓑ $f\left(2\right)$.

Solution:

ⓐ $f(0)=0$ ⓑ $f(2)=2$

Exercise:

Problem: For the function
$f\left(x\right)=\sqrt[4]{3x^3}$, find
ⓐ $f\left(0\right)$
ⓑ $f\left(3\right)$.

Exercise:

Problem: For the function
$g\left(x\right)=\sqrt[4]{4-4x}$, find
ⓐ $g\left(1\right)$
ⓑ $g\left(-3\right)$.

Solution:

ⓐ $g(1)=0$ ⓑ $g(-3)=2$

Exercise:

Problem: For the function
$g\left(x\right)=\sqrt[4]{8-4x}$, find
ⓐ $g\left(-6\right)$
ⓑ $g\left(2\right)$.

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

Exercise:

Problem: $f\left(x\right)=\sqrt{3x-1}$

Solution:

$\left[\frac{1}{3},\infty\right)$

Exercise:

Problem: $f\left(x\right)=\sqrt{4x-2}$

Exercise:

Problem: $g\left(x\right)=\sqrt{2-3x}$

Solution:

$\left(-\infty,\frac{2}{3}\right]$

Exercise:

Problem: $g\left(x\right)=\sqrt{8-x}$

Exercise:

Problem: $h\left(x\right)=\sqrt{\frac{5}{x-2}}$

Solution:

$\left(2,\infty\right)$

Exercise:

Problem: $h\left(x\right)=\sqrt{\frac{6}{x+3}}$

Exercise:

Problem: $f\left(x\right)=\sqrt{\frac{x+3}{x-2}}$

Solution:

$\left(-\infty ,-3\right]\cup \left(2,\infty \right)$

Exercise:

Problem: $f\left(x\right)=\sqrt{\frac{x-1}{x+4}}$

Exercise:

Problem: $g\left(x\right)=\sqrt[3]{8x-1}$

Solution:

$\left(-\infty ,\infty \right)$

Exercise:

Problem: $g\left(x\right)=\sqrt[3]{6x+5}$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{4{x}^{2}-16}$

Solution:

$\left(-\infty ,\infty \right)$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{6{x}^{2}-25}$

Exercise:

Problem: $F\left(x\right)=\sqrt[4]{8x+3}$

Solution:

$\left[-\frac{3}{8}, \infty\right)$

Exercise:

Problem: $F\left(x\right)=\sqrt[4]{10-7x}$

Exercise:

Problem: $G\left(x\right)=\sqrt[5]{2x-1}$

Solution:

$\left(-\infty, \infty\right)$

Exercise:

Problem: $G\left(x\right)=\sqrt[5]{6x-3}$

Graph Radical Functions

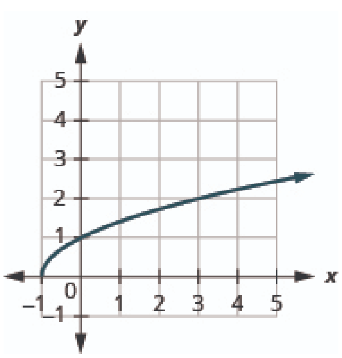
In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

Exercise:

Problem: $f\left(x\right)=\sqrt{x+1}$

Solution:

ⓐ domain: $\left[-1, \infty\right)$
ⓑ



ⓒ $[0, \infty)$

Exercise:

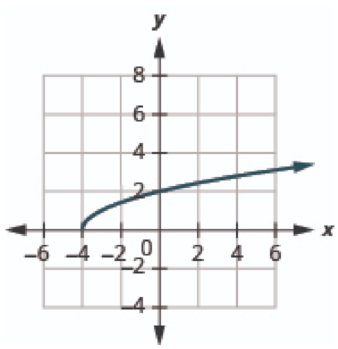
Problem: $f\left(x\right)=\sqrt{x-1}$

Exercise:

Problem: $g\left(x\right)=\sqrt{x+4}$

Solution:

ⓐ domain: $\left[-4, \infty\right)$
ⓑ



ⓒ $[0, \infty)$

Exercise:

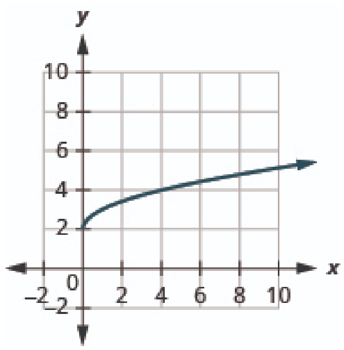
Problem: $g\left(x\right)=\sqrt{x-4}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x}+2$

Solution:

ⓐ domain: $\left[0, \infty\right)$
ⓑ



ⓒ $[2, \infty)$

Exercise:

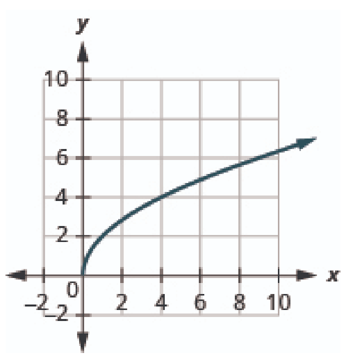
Problem: $f(x) = \sqrt{x} - 2$

Exercise:

Problem: $g(x) = 2\sqrt{x}$

Solution:

ⓐ domain: $[0, \infty)$
ⓑ



ⓒ $[0, \infty)$

Exercise:

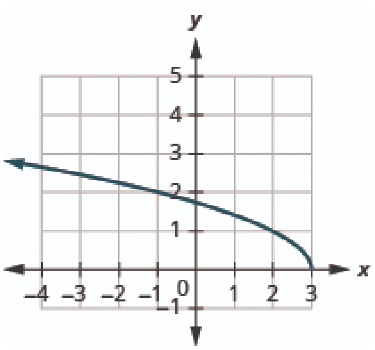
Problem: $g(x) = 3\sqrt{x}$

Exercise:

Problem: $f(x) = \sqrt{3 - x}$

Solution:

ⓐ domain: $(-\infty, 3]$
ⓑ



ⓒ $[0, \infty)$

Exercise:

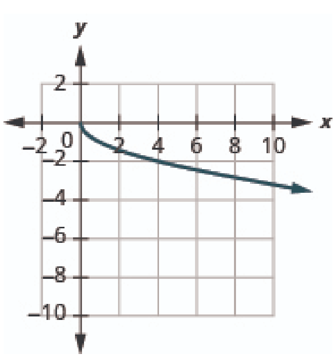
Problem: $f(x) = \sqrt{4 - x}$

Exercise:

Problem: $g(x) = -\sqrt{x}$

Solution:

ⓐ domain: $[0, \infty)$
ⓑ



ⓒ $(-\infty, 0]$

Exercise:

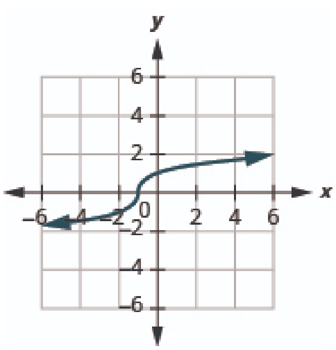
Problem: $g\left(x\right)=-\sqrt{x}+1$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{x+1}$

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ $(-\infty, \infty)$

Exercise:

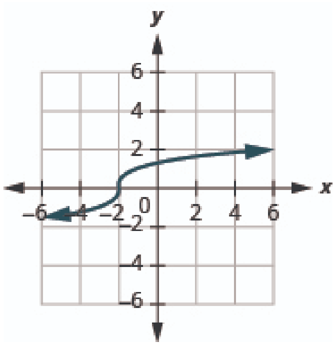
Problem: $f\left(x\right)=\sqrt[3]{x-1}$

Exercise:

Problem: $g\left(x\right)=\sqrt[3]{x+2}$

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ $(-\infty, \infty)$

Exercise:

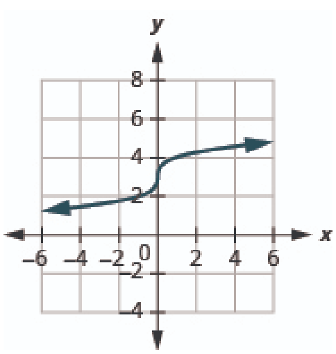
Problem: $g(x) = \sqrt[3]{x-2}$

Exercise:

Problem: $f(x) = \sqrt[3]{x} + 3$

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ $(-\infty, \infty)$

Exercise:

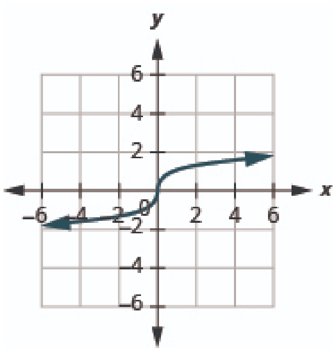
Problem: $f(x) = \sqrt[3]{x} - 3$

Exercise:

Problem: $g\left(x\right)=\sqrt[3]{x}$

Solution:

ⓐ domain: $\left(-\infty ,\infty \right)$
ⓑ



ⓒ $\left(-\infty ,\infty \right)$

Exercise:

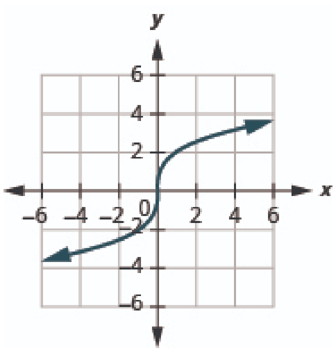
Problem: $g\left(x\right)=-\sqrt[3]{x}$

Exercise:

Problem: $f\left(x\right)=2\sqrt[3]{x}$

Solution:

ⓐ domain: $\left(-\infty ,\infty \right)$
ⓑ



ⓒ $(-\infty, \infty)$

Exercise:

Problem: $f(x) = -2\sqrt[3]{x}$

Writing Exercises

Exercise:

Problem: Explain how to find the domain of a fourth root function.

Solution:

Answers will vary.

Exercise:

Problem: Explain how to find the domain of a fifth root function.

Exercise:

Problem: Explain why $y = \sqrt[3]{x}$ is a function.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why the process of finding the domain of a radical function with an even index is different from the process when the index is odd.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
evaluate a radical function.			
find the domain of a radical function.			
graph a radical function.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

radical function
: A radical function is a function that is defined by a radical expression.

Use the Complex Number System
By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Note:
Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.5, \frac{7}{3}, 3, \sqrt{81}$, list the ⓐ rational numbers, ⓑ irrational numbers, ⓒ real numbers.
   If you missed this problem, review [link].
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review [link].
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.
   If you missed this problem, review [link].

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2 = –1$. Since all real numbers squared are positive numbers, there is no real number that equals –1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is –1.

Note:
Imaginary Unit
The **imaginary unit** i is the number whose square is –1.
Equation:

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

Note:
Square Root of a Negative Number
If b is a positive real number, then
Equation:

$$\sqrt{-b} = \sqrt{b}\,i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b} = i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b} = \sqrt{b}\,i$ is considered standard form.

Example:
Exercise:

Problem: Write each expression in terms of i and simplify if possible:

ⓐ $\sqrt{-25}$ ⓑ $\sqrt{-7}$ ⓒ $\sqrt{-12}$.

Solution:

ⓐ

	$\sqrt{-25}$
Use the definition of the square root of negative numbers.	$\sqrt{25}\,i$
Simplify.	$5i$

ⓑ

	$\sqrt{-7}$
Use the definition of the square root of negative numbers.	$\sqrt{7}i$
Simplify.	Be careful that it is clear that i is not under the radical sign.

ⓒ

	$\sqrt{-12}$
Use the definition of the square root of negative numbers.	$\sqrt{12}\,i$
Simplify $\sqrt{12}$.	$2\sqrt{3}\,i$

Note:
Exercise:

Problem: Write each expression in terms of *i* and simplify if possible:

ⓐ $\sqrt{-81}$ ⓑ $\sqrt{-5}$ ⓒ $\sqrt{-18}$.

Solution:

ⓐ $9i$ ⓑ $\sqrt{5}i$ ⓒ $3\sqrt{2}i$

Note:
Exercise:

Problem: Write each expression in terms of *i* and simplify if possible:

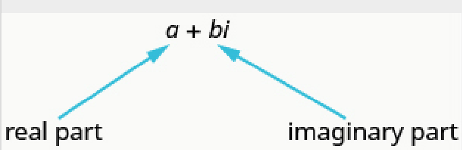
ⓐ $\sqrt{-36}$ ⓑ $\sqrt{-3}$ ⓒ $\sqrt{-27}$.

Solution:

ⓐ $6i$ ⓑ $\sqrt{3}i$ ⓒ $3\sqrt{3}i$

Now that we are familiar with the imaginary number *i*, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form *a* + *bi*, where *a*, *b* are real numbers. We call *a* the real part and *b* the imaginary part.

Note:
Complex Number
A **complex number** is of the form *a* + *bi*, where *a* and *b* are real numbers.



A complex number is in standard form when written as $a+bi$, where *a* and *b* are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

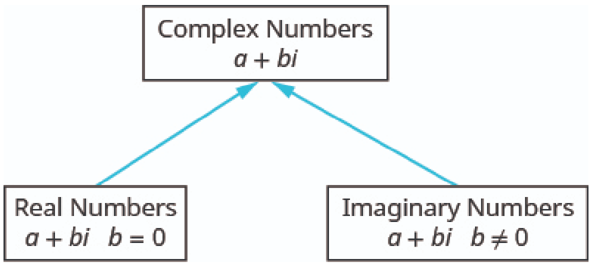
If $a = 0$, then $a + bi$ becomes $0 + bi = bi$, and is called a pure imaginary number.

We summarize this here.

	$a + bi$	
$b = 0$	$a + 0 \cdot i$ a	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi$ bi	Pure imaginary number

The standard form of a complex number is $a + bi$, so this explains why the preferred form is $\sqrt{-b} = \sqrt{b}i$ when $b > 0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.



Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example:
Exercise:

Problem: Add: $\sqrt{-12} + \sqrt{-27}$.

Solution:

	$\sqrt{-12}+\sqrt{-27}$
Use the definition of the square root of negative numbers.	$\sqrt{12}\,i+\sqrt{27}\,i$
Simplify the square roots.	$2\sqrt{3}\,i+3\sqrt{3}\,i$
Add.	$5\sqrt{3}\,i$

Note:
Exercise:

Problem: Add: $\sqrt{-8}+\sqrt{-32}$.

Solution:

$6\sqrt{2}i$

Note:
Exercise:

Problem: Add: $\sqrt{-27}+\sqrt{-48}$.

Solution:

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example:
Exercise:

Problem: Simplify: ⓐ $(4-3i)+(5+6i)$ ⓑ $(2-5i)-(5-2i)$.

Solution:

ⓐ

	$(4-3i)+(5+6i)$
Use the Associative Property to put the real parts and the imaginary parts together.	$(4+5)+(-3i+6i)$
Simplify.	$9+3i$

ⓑ

	$(2-5i)-(5-2i)$
Distribute.	$2-5i-5+2i$
Use the Associative Property to put the real parts and the imaginary parts together.	$2-5-5i+2i$
Simplify.	$-3-3i$

Note:
Exercise:

Problem: Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

Solution:

ⓐ $6+5i$ ⓑ $6-3i$

Note:
Exercise:

Problem: Simplify: ⓐ $(3-2i)+(-5-4i)$ ⓑ $(4+3i)-(2-6i)$.

Solution:

ⓐ $-2-6i$ ⓑ $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example:
Exercise:

Problem: Multiply: $2i\left(7-5i\right)$.

Solution:

	$2i\left(7-5i\right)$
Distribute.	$14i-10i^2$
Simplify i^2.	$14i-10\left(-1\right)$
Multiply.	$14i+10$
Write in standard form.	$10+14i$

Note:
Exercise:

Problem: Multiply: $4i\left(5-3i\right)$.

Solution:

$12+20i$

Note:
Exercise:

Problem: Multiply: $-3i\left(2+4i\right)$.

Solution:

$12+6i$

In the next example, we multiply the binomials using the Distributive Property or FOIL.

Example:
Exercise:

Problem: Multiply: $\left(3+2i\right)\left(4-3i\right)$.

Solution:

	$(3+2i)(4-3i)$
Use FOIL.	$12-9i+8i-6i^2$
Simplify i^2 and combine like terms.	$12-i-6(-1)$
Multiply.	$12-i+6$
Combine the real parts.	$18-i$

Note:
Exercise:

Problem: Multiply: $(5-3i)(-1-2i)$.

Solution:

$-11-7i$

Note:
Exercise:

Problem: Multiply: $(-4-3i)(2+i)$.

Solution:

$-5-10i$

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

Example:
Exercise:

Problem: Multiply: $(3+2i)^2$

Solution:

	$\begin{pmatrix} a+b \\ 3+2i \end{pmatrix}^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2 = a^2 + 2ab + b^2$.	$a^2 + 2 \quad a \quad b + b^2$ $3^2 + 2 \cdot 3 \cdot 2i + (2i)^2$
Simplify.	$9 + 12i + 4i^2$
Simplify i^2.	$9 + 12i + 4(-1)$
Simplify.	$5 + 12i$

Note:
Exercise:

Problem: Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Solution:

$-21 - 20i$

Note:
Exercise:

Problem: Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Solution:

$9 - 40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b} = \sqrt{b}i$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Example:
Exercise:

Problem: Multiply: $\sqrt{-36} \cdot \sqrt{-4}$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

	$\sqrt{-36} \cdot \sqrt{-4}$
Write as complex numbers using $\sqrt{-b} = \sqrt{b}i$.	$\sqrt{36}\, i \cdot \sqrt{4}\, i$
Simplify.	$6i \cdot 2i$
Multiply.	$12i^2$
Simplify i^2 and multiply.	-12

Note:
Exercise:

Problem: Multiply: $\sqrt{-49} \cdot \sqrt{-4}$.

Solution:

-14

Note:
Exercise:

Problem: Multiply: $\sqrt{-36} \cdot \sqrt{-81}$.

Solution:

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example:
Exercise:

Problem: Multiply: $\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

	$\left(3-\sqrt{-12}\right)\left(5+\sqrt{-27}\right)$
Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$\left(3-2\sqrt{3}\,i\right)\left(5+3\sqrt{3}\,i\right)$
Use FOIL.	$15+9\sqrt{3}\,i-10\sqrt{3}\,i-6\cdot 3i^2$
Combine like terms and simplify i^2.	$15-\sqrt{3}\,i-6\cdot(-3)$
Multiply and combine like terms.	$33-\sqrt{3}\,i$

Note:
Exercise:

Problem: Multiply: $\left(4-\sqrt{-12}\right)\left(3-\sqrt{-48}\right)$.

Solution:

$-12-22\sqrt{3}i$

Note:
Exercise:

Problem: Multiply: $\left(-2+\sqrt{-8}\right)\left(3-\sqrt{-18}\right)$.

Solution:

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Note:
Complex Conjugate Pair
A **complex conjugate pair** is of the form $a + bi$,$a - bi$.

We will multiply a complex conjugate pair in the next example.

Example:
Exercise:

Problem: Multiply: $(3 - 2i)(3 + 2i)$.

Solution:

	$(3 - 2i)(3 + 2i)$
Use FOIL.	$9 + 6i - 6i - 4i^2$
Combine like terms and simplify i^2.	$9 - 4(-1)$
Multiply and combine like terms.	13

Note:
Exercise:

Problem: Multiply: $(4 - 3i) \cdot (4 + 3i)$.

Solution:

25

Note:
Exercise:

Problem: Multiply: $(-2 + 5i) \cdot (-2 - 5i)$.

Solution:

29

From our study of polynomials, we know the product of conjugates is always of the form $(a - b)(a + b) = a^2 - b^2$. The result is called a difference of squares. We can multiply a complex

conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$\left(\overset{a}{3} \overset{-}{-} \overset{b}{2i}\right)\left(\overset{a}{3} \overset{+}{+} \overset{b}{2i}\right)$$

$$\overset{a^2}{(3)^2} \overset{-}{-} \overset{b^2}{(2i)^2}$$

$$9 - 4i^2$$

$$9 - 4(-1)$$

$$13$$

Notice this is the same result we found in [link].

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

Equation:

$$\begin{gathered}(a-bi)\,(a+bi)\\ a^2-(bi)^2\\ a^2-b^2i^2\\ a^2-b^2\,(-1)\\ a^2+b^2\end{gathered}$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)\,(a+bi)=a^2+b^2$

Note:
Product of Complex Conjugates
If a and b are real numbers, then
Equation:

$$(a-bi)\,(a+bi)=a^2+b^2$$

Example:
Exercise:

Problem: Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

	$(a-b)(a+b)$ $(8-2i)(8+2i)$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Note:
Exercise:

Problem: Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Solution:

109

Note:
Exercise:

Problem: Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Solution:

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example:

How to Divide Complex Numbers
Exercise:

Problem: Divide: $\frac{4+3i}{3-4i}$.

Solution:

Step 1. Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3. Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator.	$\frac{12+16i+9i+12i^2}{9+16}$
	Combine like terms.	$\frac{12+25i-12}{25}$
	Simplify.	$\frac{25i}{25}$
	Write the result in standard form.	i

Note:
Exercise:

Problem: Divide: $\frac{2+5i}{5-2i}$.

Solution:

i

Note:
Exercise:

Problem: Divide: $\frac{1+6i}{6-i}$.

Solution:

i

We summarize the steps here.

Note:
How to divide complex numbers.

Write both the numerator and denominator in standard form.
Multiply the numerator and denominator by the complex conjugate of the denominator.
Simplify and write the result in standard form.

Example:
Exercise:

Problem: Divide, writing the answer in standard form: $\frac{-3}{5+2i}$.

Solution:

	$\frac{-3}{5+2i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$\frac{-3(5-2i)}{(5+2i)(5-2i)}$
Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.	$\frac{-15+6i}{5^2+2^2}$
Simplify.	$\frac{-15+6i}{29}$
Write in standard form.	$-\frac{15}{29}+\frac{6}{29}i$

Note:
Exercise:

Problem: Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Solution:

$\frac{4}{17}+\frac{16}{17}i$

Note:
Exercise:

Problem: Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Solution:

$\frac{2}{5} + \frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example:
Exercise:

Problem: Divide: $\frac{5+3i}{4i}$.

Solution:

	$\frac{5+3i}{4i}$
Write the denominator in standard form.	$\frac{5+3i}{0+4i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$
Simplify.	$\frac{(5+3i)(-4i)}{(4i)(-4i)}$
Multiply.	$\frac{-20i-12i^2}{-16i^2}$
Simplify the i^2.	$\frac{-20i+12}{16}$
Rewrite in standard form.	$\frac{12}{16} - \frac{20}{16}i$
Simplify the fractions.	$\frac{3}{4} - \frac{5}{4}i$

Note:
Exercise:

Problem: Divide: $\frac{3+3i}{2i}$.

Solution:

$\frac{3}{2} - \frac{3}{2}i$

Note:
Exercise:

Problem: Divide: $\frac{2+4i}{5i}$.

Solution:

$\frac{4}{5}-\frac{2}{5}i$

Simplify Powers of *i*

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.
Equation:

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2 \cdot i & i^2 \cdot i^2 \\ & & -1 \cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4 \cdot i & i^4 \cdot i^2 & i^4 \cdot i^3 & i^4 \cdot i^4 \\ 1 \cdot i & 1 \cdot i^2 & 1 \cdot i^3 & 1 \cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.
Equation:

$$\begin{array}{lllllll} i^1 & = & i & \qquad & i^5 & = & i \\ i^2 & = & -1 & & i^6 & = & -1 \\ i^3 & = & -i & & i^7 & = & -i \\ i^4 & = & 1 & & i^8 & = & 1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4 = 1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n = \left(i^4\right)^q \cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57 = 4 \cdot 14 + 1$. So we write $i^{57} = \left(1^4\right)^{14} \cdot i^1$ and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array} \qquad \begin{array}{c} i^{57} \\ (1^4)^{14} \cdot i^1 \\ 1 \cdot i \\ i \end{array}$$

Example:
Exercise:

Problem: Simplify: i^{86}.

Solution:

	i^{86}
Divide 86 by 4 and rewrite i^{86} in the $i^n = \left(i^4\right)^q \cdot i^r$ form.	$\left(1^4\right)^{21} \cdot i^2$
$\begin{array}{r} 21 \\ 4\overline{)86} \\ \underline{8} \\ 6 \\ \underline{4} \\ 2 \end{array}$	
Simplify.	$(1)^{21} \cdot (-1)$
Simplify.	-1

Note:
Exercise:

Problem: Simplify: i^{75}.

Solution:

$-i$

Note:
Exercise:

Problem: Simplify: i^{92}.

Solution:

1

Note:
Access these online resources for additional instruction and practice with the complex number system.

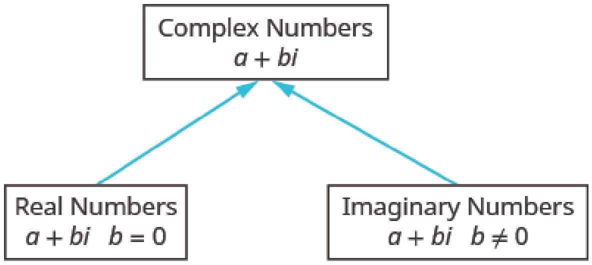
- Expressing Square Roots of Negative Numbers with i
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of i

Key Concepts

- **Square Root of a Negative Number**
    - If b is a positive real number, then $\sqrt{-b} = \sqrt{b}i$

	$a + bi$	
$b = 0$	$a + 0 \cdot i$ a	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi$ bi	Pure imaginary number

- A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.



- **Product of Complex Conjugates**
  - If a, b are real numbers, then
    $(a-bi)(a+bi)=a^2+b^2$
- **How to Divide Complex Numbers**

  Write both the numerator and denominator in standard form.
  Multiply the numerator and denominator by the complex conjugate of the denominator.
  Simplify and write the result in standard form.

Section Exercises

Practice Makes Perfect

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

Exercise:

Problem: ⓐ $\sqrt{-16}$ ⓑ $\sqrt{-11}$ ⓒ $\sqrt{-8}$

Solution:

ⓐ $4i$ ⓑ $\sqrt{11}i$ ⓒ $2\sqrt{2}i$

Exercise:

Problem: ⓐ $\sqrt{-121}$ ⓑ $\sqrt{-1}$ ⓒ $\sqrt{-20}$

Exercise:

Problem: ⓐ $\sqrt{-100}$ ⓑ $\sqrt{-13}$ ⓒ $\sqrt{-45}$

Solution:

ⓐ $10i$ ⓑ $\sqrt{13}i$ ⓒ $3\sqrt{5}i$

Exercise:

ⓐ $\sqrt{-49}$
ⓑ $\sqrt{-15}$
Problem: ⓒ $\sqrt{-75}$

Add or Subtract Complex Numbers In the following exercises, add or subtract.

Exercise:

Problem: $\sqrt{-75}+\sqrt{-48}$

Solution:

$9\sqrt{3}i$

Exercise:

Problem: $\sqrt{-12}+\sqrt{-75}$

Exercise:

Problem: $\sqrt{-50}+\sqrt{-18}$

Solution:

$8\sqrt{2}i$

Exercise:

Problem: $\sqrt{-72}+\sqrt{-8}$

Exercise:

Problem: $(1+3i)+(7+4i)$

Solution:

$8+7i$

Exercise:

Problem: $(6+2i)+(3-4i)$

Exercise:

Problem: $(8-i)+(6+3i)$

Solution:

$14+2i$

Exercise:

Problem: $(7-4i)+(-2-6i)$

Exercise:

Problem: $(1-4i)-(3-6i)$

Solution:

$-2+2i$

Exercise:

Problem: $(8-4i)-(3+7i)$

Exercise:

Problem: $(6+i)-(-2-4i)$

Solution:

$8+5i$

Exercise:

Problem: $(-2+5i)-(-5+6i)$

Exercise:

Problem: $\left(5-\sqrt{-36}\right)+\left(2-\sqrt{-49}\right)$

Solution:

$7-13i$

Exercise:

Problem: $\left(-3+\sqrt{-64}\right)+\left(5-\sqrt{-16}\right)$

Exercise:

Problem: $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

Solution:

$25-2\sqrt{2}i$

Exercise:

Problem: $\left(-5+\sqrt{-27}\right)-\left(-4-\sqrt{-48}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

Exercise:

Problem: $4i\left(5-3i\right)$

Solution:

$12+20i$

Exercise:

Problem: $2i\left(-3+4i\right)$

Exercise:

Problem: $-6i\left(-3-2i\right)$

Solution:

$-12+18i$

Exercise:

Problem: $-i\left(6+5i\right)$

Exercise:

Problem: $\left(4+3i\right)\left(-5+6i\right)$

Solution:

$-38++9i$

Exercise:

Problem: $\left(-2-5i\right)\left(-4+3i\right)$

Exercise:

Problem: $\left(-3+3i\right)\left(-2-7i\right)$

Solution:

$27+15i$

Exercise:

Problem: $(-6-2i)(-3-5i)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

Exercise:

Problem: $(3+4i)^2$

Solution:

$-7+24i$

Exercise:

Problem: $(-1+5i)^2$

Exercise:

Problem: $(-2-3i)^2$

Solution:

$-5-12i$

Exercise:

Problem: $(-6-5i)^2$

In the following exercises, multiply.

Exercise:

Problem: $\sqrt{-25}\cdot\sqrt{-36}$

Solution:

-11

Exercise:

Problem: $\sqrt{-4}\cdot\sqrt{-16}$

Exercise:

Problem: $\sqrt{-9}\cdot\sqrt{-100}$

Solution:

-30

Exercise:

Problem: $\sqrt{-64} \cdot \sqrt{-9}$

Exercise:

Problem: $\left(-2-\sqrt{-27}\right)\left(4-\sqrt{-48}\right)$

Solution:

$-44+4\sqrt{3}i$

Exercise:

Problem: $\left(5-\sqrt{-12}\right)\left(-3+\sqrt{-75}\right)$

Exercise:

Problem: $\left(2+\sqrt{-8}\right)\left(-4+\sqrt{-18}\right)$

Solution:

$-20-2\sqrt{2}i$

Exercise:

Problem: $\left(5+\sqrt{-18}\right)\left(-2-\sqrt{-50}\right)$

Exercise:

Problem: $(2-i)\,(2+i)$

Solution:

5

Exercise:

Problem: $(4-5i)\,(4+5i)$

Exercise:

Problem: $(7-2i)\,(7+2i)$

Solution:

53

Exercise:

Problem: $(-3-8i)(-3+8i)$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.
Exercise:

Problem: $(7-i)(7+i)$

Solution:

50

Exercise:

Problem: $(6-5i)(6+5i)$

Exercise:

Problem: $(9-2i)(9+2i)$

Solution:

85

Exercise:

Problem: $(-3-4i)(-3+4i)$

Divide Complex Numbers

In the following exercises, divide.
Exercise:

Problem: $\frac{3+4i}{4-3i}$

Solution:

i

Exercise:

Problem: $\frac{5-2i}{2+5i}$

Exercise:

Problem: $\frac{2+i}{3-4i}$

Solution:

$\frac{2}{25}+\frac{11}{25}i$

Exercise:

Problem: $\frac{3-2i}{6+i}$

Exercise:

Problem: $\frac{3}{2-3i}$

Solution:

$\frac{6}{13} + \frac{9}{13}i$

Exercise:

Problem: $\frac{2}{4-5i}$

Exercise:

Problem: $\frac{-4}{3-2i}$

Solution:

$-\frac{12}{13} - \frac{8}{13}i$

Exercise:

Problem: $\frac{-1}{3+2i}$

Exercise:

Problem: $\frac{1+4i}{3i}$

Solution:

$\frac{4}{3} - \frac{1}{3}i$

Exercise:

Problem: $\frac{4+3i}{7i}$

Exercise:

Problem: $\frac{-2-3i}{4i}$

Solution:

$-\frac{3}{4} + \frac{1}{2}i$

Exercise:

Problem: $\frac{-3-5i}{2i}$

Simplify Powers of *i*

In the following exercises, simplify.

Exercise:

Problem: i^{41}

Solution:

i

Exercise:

Problem: i^{39}

Exercise:

Problem: i^{66}

Solution:

-1

Exercise:

Problem: i^{48}

Exercise:

Problem: i^{128}

Solution:

1

Exercise:

Problem: i^{162}

Exercise:

Problem: i^{137}

Solution:

i

Exercise:

Problem: i^{255}

Writing Exercises

Exercise:

Problem: Explain the relationship between real numbers and complex numbers.

Solution:

Answers will vary.

Exercise:

Problem:

Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$$\begin{array}{c} \sqrt{-7} \cdot \sqrt{-7} \\ \sqrt{49} \\ 7 \end{array}$$

Exercise:

Problem: Why is $\sqrt{-64} = 8i$ but $\sqrt[3]{-64} = -4$.

Solution:

Answers will vary.

Exercise:

Problem: Explain how dividing complex numbers is similar to rationalizing a denominator.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Chapter Review Exercises

Simplify Expressions with Roots

Simplify Expressions with Roots

In the following exercises, simplify.
Exercise:

Problem: ⓐ $\sqrt{225}$ ⓑ $-\sqrt{16}$

Solution:

ⓐ 15 ⓑ -4

Exercise:

Problem: ⓐ $-\sqrt{169}$ ⓑ $\sqrt{-8}$

Exercise:

Problem: ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{243}$

Solution:

ⓐ 2 ⓑ 3 ⓒ 3

Exercise:

Problem: ⓐ $\sqrt[3]{-512}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.
Exercise:

Problem: ⓐ $\sqrt{68}$ ⓑ $\sqrt[3]{84}$

Solution:

ⓐ $8 < \sqrt{68} < 9$
ⓑ $4 < \sqrt[3]{84} < 5$

In the following exercises, approximate each root and round to two decimal places.
Exercise:

Problem: ⓐ $\sqrt{37}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{125}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.
Exercise:

Problem: ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[7]{b^7}$

Solution:

ⓐ a ⓑ $|b|$

Exercise:

Problem: ⓐ $\sqrt{a^{14}}$
ⓑ $\sqrt{w^{24}}$

Exercise:

Problem: ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

Solution:

ⓐ m^2 ⓑ n^4

Exercise:

Problem: ⓐ $\sqrt{121m^{20}}$
ⓑ $-\sqrt{64a^2}$

Exercise:

Problem: ⓐ $\sqrt[3]{216a^6}$ ⓑ $\sqrt[5]{32b^{20}}$

Solution:

ⓐ $6a^2$ ⓑ $2b^4$

Exercise:

Problem: ⓐ $\sqrt{144x^2y^2}$ ⓑ $\sqrt{169w^8y^{10}}$ ⓒ $\sqrt[3]{8a^{51}b^6}$

Simplify Radical Expressions

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

Exercise:

Problem: $\sqrt{125}$

Solution:

$5\sqrt{5}$

Exercise:

Problem: $\sqrt{675}$

Exercise:

Problem: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

Solution:

ⓐ $5\sqrt[3]{5}$ ⓑ $2\sqrt[6]{2}$

In the following exercises, simplify using absolute value signs as needed.

Exercise:

Problem: ⓐ $\sqrt{a^{23}}$
ⓑ $\sqrt[3]{b^8}$
ⓒ $\sqrt[8]{c^{13}}$

Exercise:

Problem: ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

Solution:

ⓐ $4\left|s^7\right|\sqrt{5s}$ ⓑ $2a\sqrt[5]{3a^2}$
ⓒ $2\left|b\right|\sqrt[6]{2b}$

Exercise:

Problem: ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

Exercise:

Problem: ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

Solution:

ⓐ -2 ⓑ not real

Exercise:

Problem: ⓐ $8+\sqrt{96}$
ⓑ $\frac{2+\sqrt{40}}{2}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

Exercise:

Problem: ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

Solution:

ⓐ $\frac{6}{7}$ ⓑ $\frac{2}{3}$ ⓒ $\frac{1}{2}$

Exercise:

Problem: ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

Exercise:

Problem: $\sqrt{\frac{300m^5}{64}}$

Solution:

$\frac{10m^2\sqrt{3m}}{8}$

Exercise:

ⓐ $\sqrt{\frac{28p^7}{q^2}}$

ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$

Problem: ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

Exercise:

ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$

ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

Problem: ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

Solution:

ⓐ $\frac{1}{2|pq|}$ ⓑ $\frac{2cd\sqrt[5]{2d^2}}{5}$

ⓒ $\frac{|mn|\sqrt[6]{2}}{2}$

Exercise:

ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

Problem: ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Simplify Rational Exponents

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.
Exercise:

Problem: ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

Solution:

ⓐ $\sqrt{r}$ ⓑ $\sqrt[3]{s}$ ⓒ $\sqrt[4]{t}$

In the following exercises, write with a rational exponent.
Exercise:

Problem: ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

In the following exercises, simplify.
Exercise:

Problem:
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

Solution:

ⓐ 5 ⓑ 3 ⓒ 2

Exercise:

Problem:
ⓐ $(-1{,}000)^{\frac{1}{3}}$
ⓑ $-1{,}000^{\frac{1}{3}}$
ⓒ $(1{,}000)^{-\frac{1}{3}}$

Exercise:

Problem:
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Solution:

ⓐ -2 ⓑ $\frac{1}{3}$ ⓒ -5

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.
Exercise:

Problem: ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

In the following exercises, simplify.
Exercise:

Problem: ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

Solution:

ⓐ 125 ⓑ $\frac{1}{27}$ ⓒ 16

Exercise:

Problem: ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.
Exercise:

Problem: ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Solution:

ⓐ 6^3 ⓑ b^9 ⓒ $\frac{1}{w}$

Exercise:

Problem: ⓐ $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
ⓑ $\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Add, Subtract and Multiply Radical Expressions

Add and Subtract Radical Expressions

In the following exercises, simplify.

Exercise:

Problem: ⓐ $7\sqrt{2} - 3\sqrt{2}$
ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

Solution:

ⓐ $4\sqrt{2}$ ⓑ $9\sqrt[3]{p}$ ⓒ $2\sqrt[3]{x}$

Exercise:

Problem: ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

Exercise:

Problem: ⓐ $\sqrt{48} + \sqrt{27}$
ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$
ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

Solution:

ⓐ $7\sqrt{3}$ ⓑ $7\sqrt[3]{2}$ ⓒ $3\sqrt[4]{5}$

Exercise:

Problem: ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

Exercise:

Problem: $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Solution:

$37y\sqrt{3}$

Multiply Radical Expressions

In the following exercises, simplify.
Exercise:

Problem: ⓐ $\left(5\sqrt{6}\right)\left(-\sqrt{12}\right)$ ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

Exercise:

Problem: ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$ ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Solution:

ⓐ $126x^2\sqrt{2}$ ⓑ $48a\sqrt[3]{a^2}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.
Exercise:

Problem: ⓐ $\sqrt{11}\left(8 + 4\sqrt{11}\right)$ ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9} + \sqrt[3]{18}\right)$

Exercise:

Problem: ⓐ $\left(3 - 2\sqrt{7}\right)\left(5 - 4\sqrt{7}\right)$ ⓑ $\left(\sqrt[3]{x} - 5\right)\left(\sqrt[3]{x} - 3\right)$

Solution:

ⓐ $71-22\sqrt{7}$
ⓑ $\sqrt[3]{x^2}-8\sqrt[3]{x}+15$

Exercise:

Problem: $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

Exercise:

Problem: ⓐ $\left(4+\sqrt{11}\right)^2$ ⓑ $\left(3-2\sqrt{5}\right)^2$

Solution:

ⓐ $27+8\sqrt{11}$ ⓑ $29-12\sqrt{5}$

Exercise:

Problem: $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

Exercise:

Problem: $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

Solution:

$\sqrt[3]{9x^2}-4$

Divide Radical Expressions

Divide Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

Exercise:

Problem: ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^{3}}}$ ⓑ $\frac{\sqrt[3]{16x^{4}y^{-2}}}{\sqrt[3]{-54x^{-2}y^{4}}}$

Solution:

ⓐ $\frac{8m^{4}}{3n^{4}}$ ⓑ $-\frac{x^{2}}{2y^{2}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.
Exercise:

Problem: ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^{2}}}$

Solution:

ⓐ $\frac{\sqrt[3]{121}}{11}$ ⓑ $\frac{\sqrt[3]{28}}{6}$ ⓒ $\frac{\sqrt[3]{9x}}{x}$

Exercise:

Problem: ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^{3}}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.
Exercise:

Problem: $\frac{7}{2-\sqrt{6}}$

Solution:

$-\frac{7\left(2+\sqrt{6}\right)}{2}$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Solution:

$\frac{\left(\sqrt{x}+2\sqrt{2}\right)^2}{x-8}$

Solve Radical Equations

Solve Radical Equations

In the following exercises, solve.

Exercise:

Problem: $\sqrt{4x-3}=7$

Exercise:

Problem: $\sqrt{5x+1}=-3$

Solution:

no solution

Exercise:

Problem: $\sqrt[3]{4x-1}=3$

Exercise:

Problem: $\sqrt{u-3}+3=u$

Solution:

$u=3, u=4$

Exercise:

Problem: $\sqrt[3]{4x+5}-2=-5$

Exercise:

Problem: $(8x+5)^{\frac{1}{3}}+2=-1$

Solution:

$x=-4$

Exercise:

Problem: $\sqrt{y+4}-y+2=0$

Exercise:

Problem: $2\sqrt{8r+1}-8=2$

Solution:

$r=3$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

Exercise:

Problem: $\sqrt{10+2c}=\sqrt{4c+16}$

Exercise:

Problem: $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

Solution:

$x=-8, x=2$

Exercise:

Problem: $\sqrt{r}+6=\sqrt{r+8}$

Exercise:

Problem: $\sqrt{x+1}-\sqrt{x-2}=1$

Solution:

$x=3$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

64.8 feet

Use Radicals in Functions

Evaluate a Radical Function

In the following exercises, evaluate each function.

Exercise:

Problem: $g\left(x\right)=\sqrt{6x+1}$, find
ⓐ $g\left(4\right)$
ⓑ $g\left(8\right)$

Exercise:

Problem: $G\left(x\right)=\sqrt{5x-1}$, find
ⓐ $G\left(5\right)$
ⓑ $G\left(2\right)$

Solution:

ⓐ $G(5)=2\sqrt{6}$ ⓑ $G(2)=3$

Exercise:

Problem: $h\left(x\right)=\sqrt[3]{x^2-4}$, find
ⓐ $h\left(-2\right)$
ⓑ $h\left(6\right)$

Exercise:

Problem: For the function
$g\left(x\right)=\sqrt[4]{4-4x}$, find
ⓐ $g\left(1\right)$
ⓑ $g\left(-3\right)$

Solution:

ⓐ $g(1)=0$ ⓑ $g(-3)=2$

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

Exercise:

Problem: $g(x) = \sqrt{2 - 3x}$

Exercise:

Problem: $F(x) = \sqrt{\frac{x+3}{x-2}}$

Solution:

$(2, \infty)$

Exercise:

Problem: $f(x) = \sqrt[3]{4x^2 - 16}$

Exercise:

Problem: $F(x) = \sqrt[4]{10 - 7x}$

Solution:

$\left[\frac{7}{10}, \infty\right)$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

Exercise:

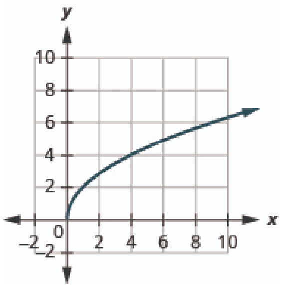
Problem: $g(x) = \sqrt{x + 4}$

Exercise:

Problem: $g(x) = 2\sqrt{x}$

Solution:

ⓐ domain: $[0, \infty)$
ⓑ



ⓒ range: $[0, \infty)$

Exercise:

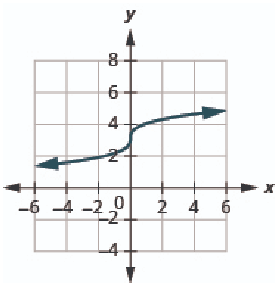
Problem: $f(x) = \sqrt[3]{x-1}$

Exercise:

Problem: $f(x) = \sqrt[3]{x} + 3$

Solution:

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ range: $(-\infty, \infty)$

Use the Complex Number System

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.
Exercise:

Problem: ⓐ $\sqrt{-100}$
ⓑ $\sqrt{-13}$
ⓒ $\sqrt{-45}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

Exercise:

Problem: $\sqrt{-50}+\sqrt{-18}$

Solution:

$8\sqrt{2}i$

Exercise:

Problem: $(8-i)+(6+3i)$

Exercise:

Problem: $(6+i)-(-2-4i)$

Solution:

$8+5i$

Exercise:

Problem: $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

Exercise:

Problem: $(-2-5i)\,(-4+3i)$

Solution:

$23+14i$

Exercise:

Problem: $-6i\,(-3-2i)$

Exercise:

Problem: $\sqrt{-4}\cdot\sqrt{-16}$

Solution:

-6

Exercise:

Problem: $\left(5-\sqrt{-12}\right)\left(-3+\sqrt{-75}\right)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.
Exercise:

Problem: $(-2-3i)^2$

Solution:

$-5-12i$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.
Exercise:

Problem: $(9-2i)(9+2i)$

Divide Complex Numbers

In the following exercises, divide.
Exercise:

Problem: $\frac{2+i}{3-4i}$

Solution:

$\frac{2}{25}+\frac{11}{25}i$

Exercise:

Problem: $\frac{-4}{3-2i}$

Simplify Powers of *i*

In the following exercises, simplify.
Exercise:

Problem: i^{48}

Solution:

1

Exercise:

Problem: i^{255}

Practice Test

In the following exercises, simplify using absolute values as necessary.

Exercise:

Problem: $\sqrt[3]{125x^9}$

Solution:

$5x^3$

Exercise:

Problem: $\sqrt{169x^8y^6}$

Exercise:

Problem: $\sqrt[3]{72x^8y^4}$

Solution:

$2x^2y\sqrt[3]{9x^2y}$

Exercise:

Problem: $\sqrt{\frac{45x^3y^4}{180x^5y^2}}$

In the following exercises, simplify. Assume all variables are positive.

Exercise:

Problem: ⓐ $216^{-\frac{1}{4}}$ ⓑ $-49^{\frac{3}{2}}$

Solution:

ⓐ $\frac{1}{4}$ ⓑ -343

Exercise:

Problem: $\sqrt{-45}$

Exercise:

Problem: $\frac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$

Solution:

$x^{\frac{7}{4}}$

Exercise:

Problem: $\left(\frac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}} \right)^{\frac{1}{3}}$

Exercise:

Problem: $\sqrt{48x^5} - \sqrt{75x^5}$

Solution:

$-x^2\sqrt{3x}$

Exercise:

Problem: $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$

Exercise:

Problem: $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$

Solution:

$36x^4\sqrt{2}$

Exercise:

Problem: $\sqrt[3]{4} \left(\sqrt[3]{16} - \sqrt[3]{6} \right)$

Exercise:

Problem: $\left(4 - 3\sqrt{3}\right) \left(5 + 2\sqrt{3}\right)$

Solution:

$2 - 7\sqrt{3}$

Exercise:

Problem: $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

Exercise:

Problem: $\frac{\sqrt{245xy^{-4}}}{\sqrt{45x^{-4}y^3}}$

Solution:

$\frac{7x^5}{3y^7}$

Exercise:

Problem: $\frac{1}{\sqrt[3]{5}}$

Exercise:

Problem: $\frac{3}{2+\sqrt{3}}$

Solution:

$3\left(2-\sqrt{3}\right)$

Exercise:

Problem: $\sqrt{-4}\cdot\sqrt{-9}$

Exercise:

Problem: $-4i\left(-2-3i\right)$

Solution:

$-12+8i$

Exercise:

Problem: $\frac{4+i}{3-2i}$

Exercise:

Problem: i^{172}

Solution:

$-i$

In the following exercises, solve.

Exercise:

Problem: $\sqrt{2x+5}+8=6$

Exercise:

Problem: $\sqrt{x+5}+1=x$

Solution:

$x=4$

Exercise:

Problem: $\sqrt[3]{2x^2 - 6x - 23} = \sqrt[3]{x^2 - 3x + 5}$

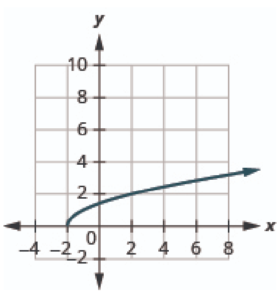
In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

Exercise:

Problem: $g(x) = \sqrt{x + 2}$

Solution:

ⓐ domain: $[-2, \infty)$
ⓑ



ⓒ range: $[0, \infty)$

Glossary

complex conjugate pair
: A complex conjugate pair is of the form *a* + *bi*, *a* – *bi*.

complex number
: A complex number is of the form *a* + *bi*, where *a* and *b* are real numbers. We call *a* the real part and *b* the imaginary part.

complex number system
: The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit
: The imaginary unit i is the number whose square is –1. $i^2 = -1$ or $i = \sqrt{-1}$.

standard form
: A complex number is in standard form when written as $a + bi$, where *a*, *b* are real numbers.

Introduction
class="introduction"

Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: “intographics”/Pixabay)

Blink your eyes. You’ve taken a photo. That’s what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

Solve Quadratic Equations Using the Square Root Property
By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x–h)^2 = k$ using the Square Root Property

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review [link].
2. Simplify: $\sqrt{\frac{32}{5}}$.
   If you missed this problem, review [link].
3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review [link].

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$	
Put the equation in standard form.	$x^2 - 9 = 0$	
Factor the difference of squares.	$(x-3)(x+3) = 0$	
Use the Zero Product Property.	$x - 3 = 0$	$x - 3 = 0$
Solve each equation.	$x = 3$	$x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Note:
Square Root Property
If $x^2 = k$, then
Equation:

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{array}{lrcl} & x^2 & = & 9 \\ \text{Use the Square Root Property.} & x & = & \pm\sqrt{9} \\ & x & = & \pm 3 \\ & \multicolumn{3}{c}{\text{So } x = 3 \text{ or } x = -3.} \end{array}$$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$\begin{array}{lc} & x^2 = 7 \\ \text{Use the Square Root Property.} & x = \sqrt{7}, \quad x = -\sqrt{7} \end{array}$$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example:
How to solve a Quadratic Equation of the form $ax^2 = k$ Using the Square Root Property
Exercise:

Problem: Solve: $x^2 - 50 = 0$.

Solution:

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2. Use Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$
Step 3. Simplify the radical.	 Rewrite to show two solutions.	$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2}, x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 ✓$ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 ✓$

Note:
Exercise:

Problem: Solve: $x^2 - 48 = 0$.

Solution:

$x = 4\sqrt{3}, x = -4\sqrt{3}$

Note:
Exercise:

Problem: Solve: $y^2 - 27 = 0$.

Solution:

$y = 3\sqrt{3}, y = -3\sqrt{3}$

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

Note:
Solve a quadratic equation using the square root property.

Isolate the quadratic term and make its coefficient one.
Use Square Root Property.
Simplify the radical.
Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example:
Exercise:

Problem: Solve: $3z^2 = 108$.

Solution:

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$

Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $\begin{array}{ll} 3z^2 = 108 & 3z^2 = 108 \\ 3(6)^2 \stackrel{?}{=} 108 & 3(-6)^2 \stackrel{?}{=} 108 \\ 3(36) \stackrel{?}{=} 108 & 3(36) \stackrel{?}{=} 108 \\ 108 = 108 \checkmark & 108 = 108 \checkmark \end{array}$	

Note:
Exercise:

Problem: Solve: $2x^2 = 98$.

Solution:

$x = 7, x = -7$

Note:
Exercise:

Problem: Solve: $5m^2 = 80$.

Solution:

$m = 4, m = -4$

The Square Root Property states ‘If $x^2 = k$,’ What will happen if $k < 0$? This will be the case in the next example.

Example:
Exercise:

Problem: Solve: $x^2 + 72 = 0$.

Solution:

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}\,i$
Simplify the radical.	$x = \pm 6\sqrt{2}\,i$
Rewrite to show two solutions.	$x = 6\sqrt{2}\,i, \;\; x = -6\sqrt{2}\,i$
Check the solutions: $x^2 + 72 = 0$ $(6\sqrt{2}\,i)^2 + 72 \stackrel{?}{=} 0$ $6^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^2 + 72 = 0$ $(6\sqrt{2}\,i)^2 + 72 \stackrel{?}{=} 0$ $(-6)^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0 \checkmark$	

Note:
Exercise:

Problem: Solve: $c^2 + 12 = 0$.

Solution:

$c = 2\sqrt{3}i, \;\; c = -2\sqrt{3}i$

Note:
Exercise:

Problem: Solve: $q^2 + 24 = 0$.

Solution:

$c = 2\sqrt{6}i,\ \ c = -2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Example:
Exercise:

Problem: Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$
Simplify.	$u^2 = 18$
Use the Square Root Property.	$u = \pm\sqrt{18}$
Simplify the radical.	

	$u = \pm\sqrt{9 \cdot 2}$
Simplify.	$u = \pm 3\sqrt{2}$
Rewrite to show two solutions.	$u = 3\sqrt{2}, \quad u = -3\sqrt{2}$
Check: $$\begin{array}{rr} \frac{2}{3}u^2 + 5 = 17 & \frac{2}{3}u^2 + 5 = 17 \\ \frac{2}{3}\left(3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17 & \frac{2}{3}\left(-3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17 \\ \frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17 & \frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17 \\ 12 + 5 \stackrel{?}{=} 17 & 12 + 5 \stackrel{?}{=} 17 \\ 17 = 17 \checkmark & 17 = 17 \checkmark \end{array}$$	

Note:
Exercise:

Problem: Solve: $\frac{1}{2}x^2 + 4 = 24.$

Solution:

$x = 2\sqrt{10}, \;\; x = -2\sqrt{10}$

Note:
Exercise:

Problem: Solve: $\frac{3}{4}y^2 - 3 = 18.$

Solution:

$y = 2\sqrt{7}, \;\; y = -2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

Example:
Exercise:

Problem: Solve: $2x^2 - 8 = 41$.

Solution:

	$2x^2 - 8 = 41$
Isolate the quadratic term.	$2x^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2} = \frac{49}{2}$
Simplify.	$x^2 = \frac{49}{2}$
Use the Square Root Property.	$x = \pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm\frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$
Simplify.	$x = \pm\frac{7\sqrt{2}}{2}$

Rewrite to show two solutions.	$x=\frac{7\sqrt{2}}{2},\quad x=-\frac{7\sqrt{2}}{2}$
Check: We leave the check for you.	

Note:
Exercise:

Problem: Solve: $5r^2-2=34$.

Solution:

$r=\frac{6\sqrt{5}}{5},\quad r=-\frac{6\sqrt{5}}{5}$

Note:
Exercise:

Problem: Solve: $3t^2+6=70$.

Solution:

$t=\frac{8\sqrt{3}}{3},\quad t=-\frac{8\sqrt{3}}{3}$

Solve Quadratic Equations of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x-h)^2=k$ as well. Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$ax^2=k \qquad a(x-h)^2=k$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

Example:
Exercise:

Problem: Solve: $4(y-7)^2 = 48$.

Solution:

	$4(y-7)^2 = 48$
Divide both sides by the coefficient 4.	$(y-7)^2 = 12$
Use the Square Root Property on the binomial	$y - 7 = \pm\sqrt{12}$
Simplify the radical.	$y - 7 = \pm 2\sqrt{3}$
Solve for y.	$y = 7 \pm 2\sqrt{3}$
Rewrite to show two solutions.	$y = 7 + 2\sqrt{3},\ y = 7 - 2\sqrt{3}$

Check:

$$\begin{array}{rr} 4(y-7)^2 = 48 & 4(y-7)^2 = 48 \\ 4\left(7+2\sqrt{3}-7\right)^2 \stackrel{?}{=} 48 & 4\left(7-2\sqrt{3}-7\right)^2 \stackrel{?}{=} 48 \\ 4\left(2\sqrt{3}\right)^2 \stackrel{?}{=} 48 & 4\left(-2\sqrt{3}\right)^2 \stackrel{?}{=} 48 \\ 4(12) \stackrel{?}{=} 48 & 4(12) \stackrel{?}{=} 48 \\ 48 = 48 \checkmark & 48 = 48 \checkmark \end{array}$$

Note:
Exercise:

Problem: Solve: $3(a-3)^2 = 54$.

Solution:

$a = 3 + 3\sqrt{2}, \;\; a = 3 - 3\sqrt{2}$

Note:
Exercise:

Problem: Solve: $2(b+2)^2 = 80$.

Solution:

$b = -2 + 2\sqrt{10}, \;\; b = -2 - 2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

Example:
Exercise:

Problem: Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

Solution:

	$\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$
Use the Square Root Property.	$x - \frac{1}{3} = \pm\sqrt{\frac{5}{9}}$
Rewrite the radical as a fraction of square roots.	$x - \frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$
Simplify the radical.	$x - \frac{1}{3} = \pm\frac{\sqrt{5}}{3}$
Solve for x.	$x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$
Rewrite to show two solutions.	$x = \frac{1}{3} + \frac{\sqrt{5}}{3}, \;\; x = \frac{1}{3} - \frac{\sqrt{5}}{3}$

Check:
We leave the check for you.

Note:

Exercise:

Problem: Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Solution:

$x=\frac{1}{2}+\frac{\sqrt{5}}{2}, x=\frac{1}{2}-\frac{\sqrt{5}}{2}$

Note:
Exercise:

Problem: Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

Solution:

$y=-\frac{3}{4}+\frac{\sqrt{7}}{4},\ \ y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial term.

Example:
Exercise:

Problem: Solve: $2(x-2)^2+3=57$.

Solution:

	$2(x-2)^2+3 = 57$
Subtract 3 from both sides to isolate the binomial term.	$2(x-2)^2 = 54$
Divide both sides by 2.	$(x-2)^2 = 27$
Use the Square Root Property.	$x-2 = \pm\sqrt{27}$
Simplify the radical.	$x-2 = \pm 3\sqrt{3}$
Solve for x.	$x = 2\pm 3\sqrt{3}$
Rewrite to show two solutions.	$x=2+3\sqrt{3},\ \ x=2-3\sqrt{3}$
Check:	
We leave the check for you.	

Note:
Exercise:

Problem: Solve: $5(a-5)^2+4=104$.

Solution:

$a=5+2\sqrt{5},\;\; a=5-2\sqrt{5}$

Note:
Exercise:

Problem: Solve: $3(b+3)^2-8=88$.

Solution:

$b=-3+4\sqrt{2},\;\; b=-3-4\sqrt{2}$

Sometimes the solutions are complex numbers.

Example:
Exercise:

Problem: Solve: $(2x-3)^2=-12$.

Solution:

$$\begin{aligned} (2x-3)^2 &= -12 \\ \text{Use the Square Root Property.} \quad 2x-3 &= \pm\sqrt{-12} \\ \text{Simplify the radical.} \quad 2x-3 &= \pm 2\sqrt{3}\,i \\ \text{Add 3 to both sides.} \quad 2x &= 3 \pm 2\sqrt{3}\,i \\ \text{Divide both sides by 2.} \quad x &= \tfrac{3\pm 2\sqrt{3}\,i}{2} \\ \text{Rewrite in standard form.} \quad x &= \tfrac{3}{2} \pm \tfrac{2\sqrt{3}\,i}{2} \\ \text{Simplify.} \quad x &= \tfrac{3}{2} \pm \sqrt{3}\,i \end{aligned}$$

Rewrite to show two solutions. $x = \frac{3}{2} + \sqrt{3}\,i, \;\; x = \frac{3}{2} - \sqrt{3}\,i$

Check:

We leave the check for you.

Note:
Exercise:

Problem: Solve: $(3r+4)^2 = -8$.

Solution:

$r = -\frac{4}{3} + \frac{2\sqrt{2}i}{3}$, r $= -\frac{4}{3} - \frac{2\sqrt{2}i}{3}$

Note:
Exercise:

Problem: Solve: $(2t-8)^2 = -10$.

Solution:

$t = 4 + \frac{\sqrt{10}i}{2}$, t $= 4 - \frac{\sqrt{10}i}{2}$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example:

Exercise:

Problem: Solve: $4n^2+4n+1=16$.

Solution:

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm4$
Solve for n.	$2n=-1\pm4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm4}{2}$ $n=\frac{-1\pm4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2},\ n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2},\qquad n=-\frac{5}{2}$
Check: $4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16\checkmark$ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16\checkmark$	

Note:

Exercise:

Problem: Solve: $9m^2 - 12m + 4 = 25$.

Solution:

$m = \frac{7}{3}, \;\; m = -1$

Note:
Exercise:

Problem: Solve: $16n^2 + 40n + 25 = 4$.

Solution:

$n = -\frac{3}{4}, \;\; n = -\frac{7}{4}$

Note:
Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property
- Using the Square Root Property to Solve Quadratic Equations

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm\sqrt{k}$

How to solve a quadratic equation using the square root property.

Isolate the quadratic term and make its coefficient one.
Use Square Root Property.
Simplify the radical.
Check the solutions.

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

Exercise:

Problem: $a^2 = 49$

Solution:

$a = \pm 7$

Exercise:

Problem: $b^2 = 144$

Exercise:

Problem: $r^2 - 24 = 0$

Solution:

$r = \pm 2\sqrt{6}$

Exercise:

Problem: $t^2 - 75 = 0$

Exercise:

Problem: $u^2 - 300 = 0$

Solution:

$u = \pm 10\sqrt{3}$

Exercise:

Problem: $v^2 - 80 = 0$

Exercise:

Problem: $4m^2 = 36$

Solution:

$m = \pm 3$

Exercise:

Problem: $3n^2 = 48$

Exercise:

Problem: $\frac{4}{3}x^2 = 48$

Solution:

$x = \pm 6$

Exercise:

Problem: $\frac{5}{3}y^2 = 60$

Exercise:

Problem: $x^2 + 25 = 0$

Solution:

$x = \pm 5i$

Exercise:

Problem: $y^2 + 64 = 0$

Exercise:

Problem: $x^2 + 63 = 0$

Solution:

$x = \pm 3\sqrt{7}i$

Exercise:

Problem: $y^2 + 45 = 0$

Exercise:

Problem: $\frac{4}{3}x^2 + 2 = 110$

Solution:

$x = \pm 9$

Exercise:

Problem: $\frac{2}{3}y^2 - 8 = -2$

Exercise:

Problem: $\frac{2}{5}a^2 + 3 = 11$

Solution:

$a = \pm 2\sqrt{5}$

Exercise:

Problem: $\frac{3}{2}b^2 - 7 = 41$

Exercise:

Problem: $7p^2 + 10 = 26$

Solution:

$p = \pm \frac{4\sqrt{7}}{7}$

Exercise:

Problem: $2q^2 + 5 = 30$

Exercise:

Problem: $5y^2 - 7 = 25$

Solution:

$y = \pm \frac{4\sqrt{10}}{5}$

Exercise:

Problem: $3x^2 - 8 = 46$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

Exercise:

Problem: $(u - 6)^2 = 64$

Solution:

$u = 14, u = -2$

Exercise:

Problem: $(v + 10)^2 = 121$

Exercise:

Problem: $(m - 6)^2 = 20$

Solution:

$m = 6 \pm 2\sqrt{5}$

Exercise:

Problem: $(n + 5)^2 = 32$

Exercise:

Problem: $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

Solution:

$r = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Exercise:

Problem: $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

Exercise:

Problem: $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

Solution:

$y = -\frac{2}{3} \pm \frac{2\sqrt{2}}{9}$

Exercise:

Problem: $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

Exercise:

Problem: $(a - 7)^2 + 5 = 55$

Solution:

$a = 7 \pm 5\sqrt{2}$

Exercise:

Problem: $(b-1)^2 - 9 = 39$

Exercise:

Problem: $4(x+3)^2 - 5 = 27$

Solution:

$x = -3 \pm 2\sqrt{2}$

Exercise:

Problem: $5(x+3)^2 - 7 = 68$

Exercise:

Problem: $(5c+1)^2 = -27$

Solution:

$c = -\frac{1}{5} \pm \frac{3\sqrt{3}}{5}i$

Exercise:

Problem: $(8d-6)^2 = -24$

Exercise:

Problem: $(4x-3)^2 + 11 = -17$

Solution:

$x = \frac{3}{4} \pm \frac{\sqrt{7}}{2}i$

Exercise:

Problem: $(2y+1)^2 - 5 = -23$

Exercise:

Problem: $m^2 - 4m + 4 = 8$

Solution:

$m = 2 \pm 2\sqrt{2}$

Exercise:

Problem: $n^2 + 8n + 16 = 27$

Exercise:

Problem: $x^2 - 6x + 9 = 12$

Solution:

$x = 3 + 2\sqrt{3}, \;\; x = 3 - 2\sqrt{3}$

Exercise:

Problem: $y^2 + 12y + 36 = 32$

Exercise:

Problem: $25x^2 - 30x + 9 = 36$

Solution:

$x = -\frac{3}{5}, x = \frac{9}{5}$

Exercise:

Problem: $9y^2 + 12y + 4 = 9$

Exercise:

Problem: $36x^2 - 24x + 4 = 81$

Solution:

$x = -\frac{7}{6}, x = \frac{11}{6}$

Exercise:

Problem: $64x^2 + 144x + 81 = 25$

Mixed Practice

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $2r^2 = 32$

Solution:

$r = \pm 4$

Exercise:

Problem: $4t^2 = 16$

Exercise:

Problem: $(a-4)^2 = 28$

Solution:

$a = 4 \pm 2\sqrt{7}$

Exercise:

Problem: $(b+7)^2 = 8$

Exercise:

Problem: $9w^2 - 24w + 16 = 1$

Solution:

$w = 1, w = \frac{5}{3}$

Exercise:

Problem: $4z^2 + 4z + 1 = 49$

Exercise:

Problem: $a^2 - 18 = 0$

Solution:

$a = \pm 3\sqrt{2}$

Exercise:

Problem: $b^2 - 108 = 0$

Exercise:

Problem: $\left(p-\frac{1}{3}\right)^2=\frac{7}{9}$

Solution:

$p=\frac{1}{3}\pm\frac{\sqrt{7}}{3}$

Exercise:

Problem: $\left(q-\frac{3}{5}\right)^2=\frac{3}{4}$

Exercise:

Problem: $m^2+12=0$

Solution:

no real solution

Exercise:

Problem: $n^2+48=0.$

Exercise:

Problem: $u^2-14u+49=72$

Solution:

$u=7\pm 6\sqrt{2}$

Exercise:

Problem: $v^2+18v+81=50$

Exercise:

Problem: $(m-4)^2+3=15$

Solution:

$m=4\pm 2\sqrt{3}$

Exercise:

Problem: $(n-7)^2-8=64$

Exercise:

Problem: $(x+5)^2 = 4$

Solution:

$x = -3, x = -7$

Exercise:

Problem: $(y-4)^2 = 64$

Exercise:

Problem: $6c^2 + 4 = 29$

Solution:

$c = \pm\frac{5\sqrt{6}}{6}$

Exercise:

Problem: $2d^2 - 4 = 77$

Exercise:

Problem: $(x-6)^2 + 7 = 3$

Solution:

no real solution

Exercise:

Problem: $(y-4)^2 + 10 = 9$

Writing Exercises

Exercise:

Problem: In your own words, explain the Square Root Property.

Solution:

Answers will vary.

Exercise:

Problem:

In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2 = 16$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Solve Quadratic Equations by Completing the Square
By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Note:
Before you get started, take this readiness quiz.

1. Expand: $(x + 9)^2$.
   If you missed this problem, review [link].
2. Factor $y^2 - 14y + 49$.
   If you missed this problem, review [link].
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review [link].

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.
Equation:

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

Equation:

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

Equation:

$$\begin{array}{cc} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)(y-7) \\ x^2 + 9x + 9x + 81 & y^2 - 7y - 7y + 49 \\ x^2 + 18x + 81 & y^2 - 14y + 49 \end{array}$$

We restate the patterns here for reference.

Note:

Binomial Squares Pattern

If *a* and *b* are real numbers,

$(a+b)^2 = a^2 + 2ab + b^2$	$\underbrace{(a+b)^2}_{(\text{binomial})^2}$	$=$	$\underbrace{a^2}_{(\text{first term})^2}$	$+$	$\underbrace{2ab}_{2\cdot(\text{product of terms})}$	$+$	$\underbrace{b^2}_{(\text{second term})^2}$
$(a-b)^2 = a^2 - 2ab + b^2$	$\underbrace{(a-b)^2}_{(\text{binomial})^2}$	$=$	$\underbrace{a^2}_{(\text{first term})^2}$	$-$	$\underbrace{2ab}_{2\cdot(\text{product of terms})}$	$+$	$\underbrace{b^2}_{(\text{second term})^2}$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\begin{array}{l} \color{red}{a^2 + 2ab + b^2} \\ x^2 + 6x + \_\_ \end{array}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of $x^2 + 6x$ is a square, x^2. This tells us that $a = x$.

$$\begin{array}{l} \color{red}{a^2 + \quad 2ab \quad + b^2} \\ x^2 + 2 \cdot x \cdot b + b^2 \end{array}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\begin{array}{l} \color{red}{a^2 + \quad 2ab \quad + b^2} \\ x^2 + 2 \cdot 3 \cdot x + \_\_ \end{array}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2 = 9$.

$$\begin{array}{l} \color{red}{a^2 + 2ab + b^2} \\ x^2 + 6x + 9 \end{array}$$

We can now factor.

$$\underbrace{(x+3)^2}_{\text{}} \quad \color{red}{(a+b)^2}$$

So we found that adding 9 to $x^2 + 6x$ ‘completes the square’, and we write it as $(x + 3)^2$.

Note:
Complete a square of $x^2 + bx$.

Identify b, the coefficient of x.
Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.
Factor the perfect square trinomial, writing it as a binomial squared.

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x^2 - 26x$ ⓑ $y^2 - 9y$ ⓒ $n^2 + \frac{1}{2}n$

Solution:

ⓐ

	$x^2 - bx$ $x^2 - 26x$
The coefficient of x is -26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2 - 26x + 169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 13)^2$

ⓑ

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

Ⓒ

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $a^2 - 20a$ ⓑ $m^2 - 5m$ ⓒ $p^2 + \frac{1}{4}p$

Solution:

ⓐ $(a - 10)^2$ ⓑ $\left(b - \frac{5}{2}\right)^2$
ⓒ $\left(p + \frac{1}{8}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $b^2 - 4b$ ⓑ $n^2 + 13n$ ⓒ $q^2 - \frac{2}{3}q$

Solution:

ⓐ $(b-2)^2$ ⓑ $\left(n + \frac{13}{2}\right)^2$
ⓒ $\left(q - \frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2 + 6x = 40$

	$x^2 + 6x + \_\_ = 40 + \_\_$
	$x^2 + 6x + 9 = 40 + 9$
Add 9 to both sides to complete the square.	$(x + 3)^2 = 49$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example:
How to Solve a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square
Exercise:

Problem: Solve by completing the square: $x^2 + 8x = 48$.

Solution:

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\underbrace{x^2 + 8x}_{x^2 + bx} = \underbrace{48}_{c}$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2}\cdot 8\right)^2}{\_\_\_} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$

Step 4. Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x+4=\pm 8$ $x+4=8 \quad x+4=-8$ $x=4 \quad x=-12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$.	$x^2+8x=48$ $(4)^2+8(4)\stackrel{?}{=}48$ $16+32\stackrel{?}{=}48$ $48=48\checkmark$
	Substitute $x=-12$.	$x^2+8x=48$ $(-12)^2+8(-12)\stackrel{?}{=}48$ $144-96\stackrel{?}{=}48$ $48=48\checkmark$

Note:
Exercise:

Problem: Solve by completing the square: $x^2+4x=5$.

Solution:

$x=-5, x=-1$

Note:
Exercise:

Problem: Solve by completing the square: $y^2-10y=-9$.

Solution:

$y = 1, y = 9$

The steps to solve a quadratic equation by completing the square are listed here.

Note:
Solve a quadratic equation of the form $x^2 + bx + c = 0$ by completing the square.

Isolate the variable terms on one side and the constant terms on the other.
Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
Use the Square Root Property.
Simplify the radical and then solve the two resulting equations.
Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example:
Exercise:

Problem: Solve by completing the square: $x^2 + 4x = -21$.

Solution:

		$\overset{x^2}{x^2} + \overset{bx}{4x} = \overset{c}{-21}$
The variable terms are on the left side. Take half of 4 and square it.		$x^2 + 4x + \underset{\left(\frac{1}{2} \cdot 4\right)^2}{\underline{\qquad}} = -21$
$\left(\frac{1}{2}(4)\right)^2 = 4$		
Add 4 to both sides.		$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.		$(x + 2)^2 = -17$
Use the Square Root Property.		$x + 2 = \pm\sqrt{-17}$
Simplify using complex numbers.		$x + 2 = \pm\sqrt{17}\, i$
Subtract 2 from each side.		$x = -2 \pm \sqrt{17}\, i$

Rewrite to show two solutions.	$x = -2 + \sqrt{17}\,i, \quad x = -2 - \sqrt{17}\,i$
We leave the check to you.	

Note:
Exercise:

Problem: Solve by completing the square: $y^2 - 10y = -35$.

Solution:

$y = 5 + \sqrt{15}i, \;\; y = 5 - \sqrt{15}i$

Note:
Exercise:

Problem: Solve by completing the square: $z^2 + 8z = -19$.

Solution:

$z = -4 + \sqrt{3}i, \; \text{z} = -4 - \sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example:
Exercise:

Problem: Solve by completing the square: $y^2 - 18y = -6$.

Solution:

	$x^2 - \quad bx \quad\quad c$ $y^2 - 18y = -6$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\hspace{2cm}}} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y - 9)^2 = 75$
Use the Square Root Property.	$y - 9 = \pm\sqrt{75}$

Simplify the radical.	$y-9=\pm 5\sqrt{3}$
Solve for y.	$y=9\pm 5\sqrt{3}$

Check.

$$\begin{aligned} y^2-18y &= -6 \\ (9+5\sqrt{3})^2-18(9+5\sqrt{3}) &\stackrel{?}{=} -6 \\ 81+90\sqrt{3}+75-162+90\sqrt{3} &\stackrel{?}{=} -6 \\ -6 &= -6\ \checkmark \end{aligned}$$

$$\begin{aligned} y^2-18y &= -6 \\ (9-5\sqrt{3})^2-18(9-5\sqrt{3}) &\stackrel{?}{=} -6 \\ 81+90\sqrt{3}+75-162+90\sqrt{3} &\stackrel{?}{=} -6 \\ -6 &= -6\ \checkmark \end{aligned}$$

Another way to check this would be to use a calculator. Evaluate $y^2-18y$for both of the solutions. The answer should be -6.

Note:
Exercise:

Problem: Solve by completing the square: $x^2-16x=-16$.

Solution:

$x=8+4\sqrt{3},\ \ x=8-4\sqrt{3}$

Note:
Exercise:

Problem: Solve by completing the square: $y^2+8y=11$.

Solution:

$y = -4 + 3\sqrt{3},\ y = -4 - 3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example:
Exercise:

Problem: Solve by completing the square: $x^2 + 10x + 4 = 15$.

Solution:

	$x^2 + 10x + 4 = 15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2 + 10x = 11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2 = 25$	$x^2 - 10x + \underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\underline{\qquad\qquad}} = 11$

Add 25 to both sides.	$x^2 + 10x + 25 = 11 + 25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x + 5)^2 = 36$
Use the Square Root Property.	$x + 5 = \pm\sqrt{36}$
Simplify the radical.	$x + 5 = \pm 6$
Solve for *x*.	$x = -5 \pm 6$
Rewrite to show two solutions.	$x = -5 + 6, \quad x = -5 - 6$
Solve the equations.	$x = 1, \quad x = -11$

Check:

$$\begin{array}{rr} x^2 + 10x + 4 = 15 & x^2 + 10x + 4 = 15 \\ (1)^2 + 10(1) + 4 \stackrel{?}{=} 15 & (-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15 \\ 1 + 10 + 4 \stackrel{?}{=} 15 & 121 + 110 + 4 \stackrel{?}{=} 15 \\ 15 = 15 \checkmark & 15 = 15 \checkmark \end{array}$$

Note:
Exercise:

Problem: Solve by completing the square: $a^2 + 4a + 9 = 30$.

Solution:

$a = -7, a = 3$

Note:
Exercise:

Problem: Solve by completing the square: $b^2 + 8b - 4 = 16$.

Solution:

$b = -10, b = 2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example:
Exercise:

Problem: Solve by completing the square: $n^2 = 3n + 11$.

Solution:

	$n^2 = 3n + 11$
Subtract $3n$ to get the variable terms on the left side.	$n^2 - 3n = 11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$	$n^2 - 3n + \underline{\qquad} = 11$ $\left(\frac{1}{2} \cdot (-3)\right)^2$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for n.	$n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	

	$n = \frac{3}{2} + \frac{\sqrt{53}}{2}, \quad n = \frac{3}{2} - \frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by completing the square: $p^2 = 5p + 9$.

Solution:

$p = \frac{5}{2} + \frac{\sqrt{61}}{2},\ p = \frac{5}{2} - \frac{\sqrt{61}}{2}$

Note:
Exercise:

Problem: Solve by completing the square: $q^2 = 7q - 3$.

Solution:

$q = \frac{7}{2} + \frac{\sqrt{37}}{2}$, $\mathrm{q} = \frac{7}{2} - \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says “If $a \cdot b = 0$, then $a = 0$ or $b = 0$.” Instead, we multiply the factors

and then put the equation into standard form to solve by completing the square.

Example:
Exercise:

Problem: Solve by completing the square: $\left(x-3\right)\left(x+5\right)=9$.

Solution:

			$(x-3)(x+5)=9$
We multiply the binomials on the left.			$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.			$x^2+2x\qquad=24$
Take half of 2 and square it.			
$\left(\frac{1}{2}\cdot(2)\right)^2=1$			$x^2+2x+\underset{\left(\frac{1}{2}\cdot(2)\right)^2}{\underline{\qquad}}=24$
Add 1 to both sides.			$x^2+2x+1=24+1$
Factor the perfect square			$(x+1)^2=25$

trinomial, writing it as a binomial squared.		
Use the Square Root Property.		$x+1=\pm\sqrt{25}$
Solve for *x*.		$x=-1\pm 5$
Rewrite to show two solutions.	$x=-1+5, x=-1-5$	
Simplify.	$x=4, \qquad x=-6$	
Check: We leave the check for you!		

Note:
Exercise:

Problem: Solve by completing the square: $(c-2)(c+8)=11.$

Solution:

$c=-9, c=3$

Note:
Exercise:

Problem: Solve by completing the square: $(d-7)(d+3)=56$.

Solution:

$d=11, d=-7$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example:
Exercise:

Problem: Solve by completing the square: $3x^2-12x-15=0$.

Solution:

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x \qquad = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\underline{\qquad}} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for *x*.	$x - 2 = \pm 3$

Rewrite to show two solutions.	$x = 2 + 3, \; x = 2 - 3$
Simplify.	$x = 5, \qquad x = -1$
Check: $$\begin{array}{rr} x = 5 & x = -1 \\ 3x^2 - 12x - 15 = 0 & 3x^2 - 12x - 15 = 0 \\ 3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0 & 3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0 \\ 75 - 60 - 15 \stackrel{?}{=} 0 & 3 + 12 - 15 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$	

Note:
Exercise:

Problem: Solve by completing the square: $2m^2 + 16m + 14 = 0$.

Solution:

$m = -7, m = -1$

Note:
Exercise:

Problem: Solve by completing the square: $4n^2 - 24n - 56 = 8$.

Solution:

$n = -2, n = 8$

To complete the square, the coefficient of the x^2 must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example:
Exercise:

Problem: Solve by completing the square: $2x^2 - 3x = 20$.

Solution:

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

		$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.		$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.		$x^2 - \frac{3}{2}x = 10$

Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\underline{\qquad\qquad}} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4},\quad x = \frac{3}{4} - \frac{13}{4}$

Simplify.	$x = 4, \qquad x = -\frac{5}{2}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by completing the square: $3r^2 - 2r = 21$.

Solution:

$r = -\frac{7}{3}, r = 3$

Note:
Exercise:

Problem: Solve by completing the square: $4t^2 + 2t = 20$.

Solution:

$t = -\frac{5}{2}, t = 2$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a quadratic

equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

Note:
Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

Divide by a to make the coefficient of x^2 term 1.
Isolate the variable terms on one side and the constant terms on the other.
Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
Use the Square Root Property.
Simplify the radical and then solve the two resulting equations.
Check the solutions.

Example:
Exercise:

Problem: Solve by completing the square: $3x^2 + 2x = 4$.

Solution:

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\underline{\qquad}}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for x .	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$

Rewrite to show two solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\quad x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by completing the square: $4x^2+3x=2$.

Solution:

$x=-\frac{3}{8}+\frac{\sqrt{41}}{8},\ x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$

Note:
Exercise:

Problem: Solve by completing the square: $3y^2-10y=-5$.

Solution:

$y=\frac{5}{3}+\frac{\sqrt{10}}{3},\ y=\frac{5}{3}-\frac{\sqrt{10}}{3}$

Note:
Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

  $$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

  $$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square

  Identify b, the coefficient of x.
  Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx
  Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

  Divide by a to make the coefficient of x^2 term 1.
  Isolate the variable terms on one side and the constant terms on the other.
  Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  Factor the perfect square trinomial, writing it as a binomial squared on

the left and simplify by adding the terms on the right.
Use the Square Root Property.
Simplify the radical and then solve the two resulting equations.
Check the solutions.

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

Exercise:

Problem: ⓐ $m^2 - 24m$
ⓑ $x^2 - 11x$
ⓒ $p^2 - \frac{1}{3}p$

Solution:

ⓐ $(m - 12)^2$ ⓑ $\left(x - \frac{11}{2}\right)^2$
ⓒ $\left(p - \frac{1}{6}\right)^2$

Exercise:

Problem: ⓐ $n^2 - 16n$
ⓑ $y^2 + 15y$
ⓒ $q^2 + \frac{3}{4}q$

Exercise:

Problem: ⓐ $p^2 - 22p$
ⓑ $y^2 + 5y$
ⓒ $m^2 + \frac{2}{5}m$

Solution:

ⓐ $(p-11)^2$ ⓑ $\left(y+\frac{5}{2}\right)^2$
ⓒ $\left(m+\frac{1}{5}\right)^2$

Exercise:

Problem: ⓐ $q^2 - 6q$
ⓑ $x^2 - 7x$
ⓒ $n^2 - \frac{2}{3}n$

Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: 5. $u^2 + 2u = 3$

Solution:

$u = -3, u = 1$

Exercise:

Problem: $z^2 + 12z = -11$

Exercise:

Problem: $x^2 - 20x = 21$

Solution:

$x = -1, x = 21$

Exercise:

Problem: $y^2 - 2y = 8$

Exercise:

Problem: $m^2 + 4m = -44$

Solution:

$m = -2 \pm 2\sqrt{10}i$

Exercise:

Problem: $n^2 - 2n = -3$

Exercise:

Problem: $r^2 + 6r = -11$

Solution:

$r = -3 \pm \sqrt{2}i$

Exercise:

Problem: $t^2 - 14t = -50$

Exercise:

Problem: $a^2 - 10a = -5$

Solution:

$a = 5 \pm 2\sqrt{5}$

Exercise:

Problem: $b^2 + 6b = 41$

Exercise:

Problem: $x^2 + 5x = 2$

Solution:

$x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

Exercise:

Problem: $y^2 - 3y = 2$

Exercise:

Problem: $u^2 - 14u + 12 = 1$

Solution:

$u = 1, u = 13$

Exercise:

Problem: $z^2 + 2z - 5 = 2$

Exercise:

Problem: $r^2 - 4r - 3 = 9$

Solution:

$r = -2, r = 6$

Exercise:

Problem: $t^2 - 10t - 6 = 5$

Exercise:

Problem: $v^2 = 9v + 2$

Solution:

$v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

Exercise:

Problem: $w^2 = 5w - 1$

Exercise:

Problem: $x^2 - 5 = 10x$

Solution:

$x = 5 \pm \sqrt{30}$

Exercise:

Problem: $y^2 - 14 = 6y$

Exercise:

Problem: $(x + 6)(x - 2) = 9$

Solution:

$x = -7, x = 3$

Exercise:

Problem: $(y+9)(y+7)=80$

Exercise:

Problem: $(x+2)(x+4)=3$

Solution:

$x=-5, x=-1$

Exercise:

Problem: $(x-2)(x-6)=5$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $3m^2+30m-27=6$

Solution:

$m=-11, m=1$

Exercise:

Problem: $2x^2-14x+12=0$

Exercise:

Problem: $2n^2+4n=26$

Solution:

$n = 1 \pm 2\sqrt{3}$

Exercise:

Problem: $5x^2 + 20x = 15$

Exercise:

Problem: $2c^2 + c = 6$

Solution:

$c = -2, c = \frac{3}{2}$

Exercise:

Problem: $3d^2 - 4d = 15$

Exercise:

Problem: $2x^2 + 7x - 15 = 0$

Solution:

$x = -5, x = \frac{3}{2}$

Exercise:

Problem: $3x^2 - 14x + 8 = 0$

Exercise:

Problem: $2p^2 + 7p = 14$

Solution:

$p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

Exercise:

Problem: $3q^2 - 5q = 9$

Exercise:

Problem: $5x^2 - 3x = -10$

Solution:

$x = \frac{3}{10} \pm \frac{\sqrt{191}}{10} i$

Exercise:

Problem: $7x^2 + 4x = -3$

Writing Exercises

Exercise:

Problem: Solve the equation $x^2 + 10x = -25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

Solution:

Answers will vary.

Exercise:

Problem:

Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Solve Quadratic Equations Using the Quadratic Formula
By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Note:
Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
   If you missed this problem, review [link].
2. Simplify: $\sqrt{108}$.
   If you missed this problem, review [link].
3. Simplify: $\sqrt{50}$.
   If you missed this problem, review [link].

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

	$ax^2 + bx + c = 0 \quad a \neq 0$
Isolate the variable terms on one side.	$ax^2 + bx = -c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$
Simplify.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Use the square root property.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
	This equation is the Quadratic Formula.

Note:
Quadratic Formula
The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:
Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

Example:
How to Solve a Quadratic Equation Using the Quadratic Formula
Exercise:

Problem: Solve by using the Quadratic Formula: $2x^2 + 9x - 5 = 0$.

Solution:

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a = 2$, $b = 9$, $c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad$ $x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad$ $x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad$ $x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$. Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

Solution:

$y = 1, y = \frac{2}{3}$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $4z^2 + 2z - 6 = 0$.

Solution:

$z = 1, z = -\frac{3}{2}$

Note:
Solve a quadratic equation using the quadratic formula.

Write the quadratic equation in standard form, $ax^2+bx+c = 0$. Identify the values of $a, b,$ and c.
Write the Quadratic Formula. Then substitute in the values of $a, b,$ and c.
Simplify.
Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x =$".

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution:

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of $a,\ b,\ c$.	$a = 1,\ b = -6,\ c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of $a,\ b,\ c$.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$

	$x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, \quad x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, \quad x = \frac{2}{2}$
	$x = 5, \quad x = 1$
Check: $\begin{array}{rr} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$	

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

Solution:

$a = -3, a = 5$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

Solution:

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution:

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 2x^2 + 10x + 11 = 0 \end{array}$
Identify the values of a, b, and c.	$a = 2, b = 10, c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Then substitute in the values of *a*, *b*, and *c*.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + \sqrt{3}}{2}, \quad x = \frac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

Solution:

$m = \frac{-6+\sqrt{15}}{3}, m = \frac{-6-\sqrt{15}}{3}$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

Solution:

$n = \frac{-2+2\sqrt{6}}{5}, n = \frac{-2-2\sqrt{6}}{5}$

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

Solution:

	$3p^2 + 2p + 9 = 0$
This equation is in standard form	$\begin{matrix} ax^2 + bx + c = 0 \\ 3p^2 + 2p + 9 = 0 \end{matrix}$

Identify the values of a, b, c.	$a = 3, b = 2, c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
	$p = \frac{-2 \pm \sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p = \frac{-2 \pm \sqrt{104}\, i}{6}$
Simplify the radical.	$p = \frac{-2 \pm 2\sqrt{26}\, i}{6}$
Factor the common factor in the numerator.	$p = \frac{2\left(-1 \pm \sqrt{26}\, i\right)}{6}$
Remove the common factors.	$p = \frac{-1 \pm \sqrt{26}\, i}{3}$
Rewrite in standard $a + bi$ form.	$p = -\frac{1}{3} \pm \frac{\sqrt{26}\, i}{3}$
Write as two solutions.	

$p = -\frac{1}{3} + \frac{\sqrt{26}\,i}{3}, \quad p = -\frac{1}{3} - \frac{\sqrt{26}\,i}{3}$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $4a^2 - 2a + 8 = 0$.

Solution:

$a = \frac{1}{4} + \frac{\sqrt{31}}{4}i, \;\; a = \frac{1}{4} - \frac{\sqrt{31}}{4}i$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $5b^2 + 2b + 4 = 0$.

Solution:

$b = -\frac{1}{5} + \frac{\sqrt{19}}{5}i, \;\; b = -\frac{1}{5} - \frac{\sqrt{19}}{5}i$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $x\left(x + 6\right) + 4 = 0$.

Solution:

Our first step is to get the equation in standard form.

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c .	$x=\dfrac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\dfrac{-6\pm\sqrt{36-16}}{2}$
	$x=\dfrac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\dfrac{-6\pm 2\sqrt{5}}{2}$

Factor the common factor in the numerator.	$x = \frac{2(-3 \pm 2\sqrt{5})}{2}$
Remove the common factors.	$x = -3 \pm 2\sqrt{5}$
Write as two solutions.	$x = -3 + 2\sqrt{5}, \quad x = -3 - 2\sqrt{5}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $x\left(x+2\right)-5=0$.

Solution:

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $3y\left(y-2\right)-3=0$.

Solution:

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

	$\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$
Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the values of *a*, *b*, and *c*.	$a = 3, b = 4, c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Then substitute in the values of *a*, *b*, and *c*.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$
	$u=\frac{-4\pm\sqrt{40}}{6}$
Simplify the radical.	$u=\frac{-4\pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2(-2\pm\sqrt{10})}{6}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$
Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3},\quad u=\frac{-2-\sqrt{10}}{3}$
Check: We leave the check for you!	

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $\frac{1}{4}c^2-\frac{1}{3}c=\frac{1}{12}$.

Solution:

$c = \frac{2+\sqrt{7}}{3}, \; c = \frac{2-\sqrt{7}}{3}$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Solution:

$d = \frac{9+\sqrt{33}}{4}$, d $= \frac{9-\sqrt{33}}{4}$

Think about the equation $(x - 3)^2 = 0$. We know from the Zero Product Property that this equation has only one solution,
$x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

Example:
Exercise:

Problem: Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

	$4x^2 - 20x = -25$

Add 25 to get the equation in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 4x^2 - 20x + 25 = 0 \end{array}$
Identify the values of *a*, *b*, and *c*.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$x = \dfrac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \dfrac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \dfrac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \dfrac{20}{8}$
Simplify the fraction.	$x = \dfrac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Solution:

$r = -5$

Note:
Exercise:

Problem: Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Solution:

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Note:
Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the quantity $b^2 - 4ac$ is called the discriminant.

Let’s look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	**Discriminant** $b^2 - 4ac$	**Value of the Discriminant**	**Number and Type of solutions**
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2\,(-5)$ 121	+	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	−	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions.** $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution.** $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions.** $x = \frac{-b \pm \sqrt{-}}{2a}$

Note:

Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example:
Exercise:

Problem: Determine the number of solutions to each quadratic equation.

ⓐ $3x^2 + 7x - 9 = 0$ ⓑ $5n^2 + n + 4 = 0$ ⓒ $9y^2 - 6y + 1 = 0$.

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 3, \;\; b = 7, \; c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$
	157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

$$5n^2 + n + 4 = 0$$

The equation is in standard form, identify $a, b,$ and c. $\quad a = 5, \;\; b = 1, \;\; c = 4$

Write the discriminant. $\quad b^2 - 4ac$

Substitute in the values of $a, b,$ and c. $\quad (1)^2 - 4 \cdot 5 \cdot 4$

Simplify. $\quad 1 - 80$

$$-79$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

$$9y^2 - 6y + 1 = 0$$

The equation is in standard form, identify $a, b,$ and c. $\quad a = 9, b = -6, c = 1$

Write the discriminant. $\quad b^2 - 4ac$

Substitute in the values of $a, b,$ and c. $\quad (-6)^2 - 4 \cdot 9 \cdot 1$

Simplify. $\quad 36 - 36$

$$0$$

Since the discriminant is 0, there is 1 real solution to the equation.

Note:
Exercise:

Problem:

Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0.$

Solution:

ⓐ 2 complex solutions; ⓑ 2 real solutions; ⓒ 1 real solution

Note:
Exercise:

Problem:

Determine the number and type of solutions to each quadratic equation.

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0.$

Solution:

ⓐ 2 real solutions; ⓑ 2 complex solutions; ⓒ 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Note:
Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2 = k$ or $a(x - h)^2 = k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods

because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Note:
Identify the most appropriate method to solve a quadratic equation.

Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.

Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example:
Exercise:

Problem:

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$.

Solution:

ⓐ

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

$$\begin{array}{lrcl} & 8u^2+6u & = & 11 \\ \text{Put the equation in standard form.} & 8u^2+6u-11 & = & 0 \end{array}$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Note:
Exercise:

Problem:

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x^2+6x+8=0$ ⓑ $(n-3)^2=16$ ⓒ $5p^2-6p=9$.

Solution:

ⓐ factoring; ⓑ Square Root Property; ⓒ Quadratic Formula

Note:
Exercise:

Problem:

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a^2+3a-9=0$ ⓑ $4b^2+4b+1=0$ ⓒ $5c^2=125$.

Solution:

ⓐ Quadratic Forumula;
ⓑ Factoring or Square Root Property ⓒ Square Root Property

Note:
Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:
    Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.

  Write the quadratic equation in standard form, $ax^2+bx+c = 0$. Identify the values of a,b,c.
  Write the Quadratic Formula. Then substitute in the values of a,b,c.
  Simplify.
  Check the solutions.

- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - if $b^2 - 4ac = 0$, the equation has 1 real solution.
    - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:

    - Factoring
    - Square Root Property
    - Completing the Square
    - Quadratic Formula

- How to identify the most appropriate method to solve a quadratic equation.

  Try Factoring first. If the quadratic factors easily, this method is very quick.

Try the	**Square Root Property**	next. If the equation fits the form	$ax^2=k$ or $a(x-h)^2=k$,	it can easily be solved by using the Square Root Property.
Use the	**Quadratic Formula.**	Any other quadratic equation is best solved by using the Quadratic Formula.		

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

Exercise:

Problem: $4m^2+m-3=0$

Solution:

$m=-1, m=\frac{3}{4}$

Exercise:

Problem: $4n^2-9n+5=0$

Exercise:

Problem: $2p^2-7p+3=0$

Solution:

$p=\frac{1}{3}, p=2$

Exercise:

Problem: $3q^2 + 8q - 3 = 0$

Exercise:

Problem: $p^2 + 7p + 12 = 0$

Solution:

$p = -4, p = -3$

Exercise:

Problem: $q^2 + 3q - 18 = 0$

Exercise:

Problem: $r^2 - 8r = 33$

Solution:

$r = -3, r = 11$

Exercise:

Problem: $t^2 + 13t = -40$

Exercise:

Problem: $3u^2 + 7u - 2 = 0$

Solution:

$u = \frac{-7 \pm \sqrt{73}}{6}$

Exercise:

Problem: $2p^2 + 8p + 5 = 0$

Exercise:

Problem: $2a^2 - 6a + 3 = 0$

Solution:

$a = \frac{3 \pm \sqrt{3}}{2}$

Exercise:

Problem: $5b^2 + 2b - 4 = 0$

Exercise:

Problem: $x^2 + 8x - 4 = 0$

Solution:

$x = -4 \pm 2\sqrt{5}$

Exercise:

Problem: $y^2 + 4y - 4 = 0$

Exercise:

Problem: $3y^2 + 5y - 2 = 0$

Solution:

$y = -\frac{2}{3}, y = -1$

Exercise:

Problem: $6x^2 + 2x - 20 = 0$

Exercise:

Problem: $2x^2 + 3x + 3 = 0$

Solution:

$x = -\frac{3}{4} \pm \frac{\sqrt{15}}{4}i$

Exercise:

Problem: $2x^2 - x + 1 = 0$

Exercise:

Problem: $8x^2 - 6x + 2 = 0$

Solution:

$x = \frac{3}{8} \pm \frac{\sqrt{7}}{8}i$

Exercise:

Problem: $8x^2 - 4x + 1 = 0$

Exercise:

Problem: $(v+1)(v-5) - 4 = 0$

Solution:

$v = 2 \pm 2\sqrt{2}$

Exercise:

Problem: $(x+1)(x-3) = 2$

Exercise:

Problem: $(y+4)(y-7) = 18$

Solution:

$y = -4, y = 7$

Exercise:

Problem: $(x+2)(x+6) = 21$

Exercise:

Problem: $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

Solution:

$m=-1, m=\frac{3}{4}$

Exercise:

Problem: $\frac{1}{3}n^2+n=-\frac{1}{2}$

Exercise:

Problem: $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$

Solution:

$b=\frac{-2\pm\sqrt{11}}{6}$

Exercise:

Problem: $\frac{1}{9}c^2+\frac{2}{3}c=3$

Exercise:

Problem: $16c^2+24c+9=0$

Solution:

$c=-\frac{3}{4}$

Exercise:

Problem: $25d^2-60d+36=0$

Exercise:

Problem: $25q^2+30q+9=0$

Solution:

$q=-\frac{3}{5}$

Exercise:

Problem: $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

Exercise:

Problem: ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$

Solution:

ⓐ no real solutions ⓑ 1
ⓒ 2

Exercise:

Problem: ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$

Exercise:

Problem: ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $3v^2 - 5v - 1 = 0$

Solution:

ⓐ 1 ⓑ no real solutions
ⓒ 2

Exercise:

ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
Problem: ⓒ $7y^2 + 2y + 8 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

Exercise:

ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y+5)^2 = 12$
Problem: ⓒ $14m^2 + 3m = 11$

Solution:

ⓐ factor ⓑ square root
ⓒ Quadratic Formula

Exercise:

ⓐ $(8v+3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
Problem: ⓒ $4n^2 - 10 = 6$

Exercise:

ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
Problem: ⓒ $y^2 - 2y = 8$

Solution:

ⓐ Quadratic Formula
ⓑ square root ⓒ factor

Exercise:

Problem: ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Writing Exercises

Exercise:

Problem: Solve the equation $x^2 + 10x = 120$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

Solution:

Answers will vary.

Exercise:

Problem: Solve the equation $12y^2 + 23y = 24$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

discriminant

In the Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the discriminant.

Solve Quadratic Equations in Quadratic Form
By the end of this section, you will be able to:

- Solve equations in quadratic form

Note:
Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20.$
   If you missed this problem, review [link].
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15.$
   If you missed this problem, review [link].
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2.$
   If you missed this problem, review [link].

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$

	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u + 1)(u - 5)$
Replace u with x^2.	$(x^2 + 1)(x^2 - 5)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

Example:
How to Solve Equations in Quadratic Form
Exercise:

Problem: Solve: $6x^4 - 7x^2 + 2 = 0$

Solution:

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2} \quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \qquad x = -\frac{\sqrt{6}}{3}$

Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0 \checkmark$ We leave the other checks to you!

Note:
Exercise:

Problem: Solve: $x^4 - 6x^2 + 8 = 0$.

Solution:

$x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$

Note:
Exercise:

Problem: Solve: $x^4 - 11x^2 + 28 = 0$.

Solution:

$x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$

We summarize the steps to solve an equation in quadratic form.

Note:
Solve equations in quadratic form.

Identify a substitution that will put the equation in quadratic form.
Rewrite the equation with the substitution to put it in quadratic form.
Solve the quadratic equation for*u*.
Substitute the original variable back into the results, using the substitution.
Solve for the original variable.
Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

Example:
Exercise:

Problem: Solve: $(x-2)^2 + 7(x-2) + 12 = 0.$

Solution:

	$(x - 2)^2 + 7(x - 2) + 12 = 0$

Prepare for the substitution.	$(x-2)^2+7(x-2)+12=0$
Let $u=x-2$ and substitute.	$u^2+7u+12=0$
Solve by factoring.	$(u+3)(u+4)=0$ $u+3=0, \quad u+4=0$ $u=-3, \quad u=-4$
Replace u with $x-2$.	$x-2=-3, \quad x-2=-4$
Solve for x.	$x=-1, \quad x=-2$
Check: $x=-1$ $(x-2)^2+7(x-2)+12=0$ $(-1-2)^2+7(-1-2)+12\stackrel{?}{=}0$ $(-3)^2+7(-3)+12\stackrel{?}{=}0$ $9-21+12\stackrel{?}{=}0$ $0=0$ ✓ $x=-2$ $(x-2)^2+7(x-2)+12=0$ $(-2-2)^2+7(-2-2)+12\stackrel{?}{=}0$ $(-4)^2+7(-4)+12\stackrel{?}{=}0$ $16-28+12\stackrel{?}{=}0$ $0=0$ ✓	

Note:
Exercise:

Problem: Solve: $(x-5)^2+6(x-5)+8=0$.

Solution:

$x=3, x=1$

Note:
Exercise:

Problem: Solve: $(y-4)^2+8(y-4)+15=0$.

Solution:

$y=-1, y=1$

In the next example, we notice that $\left(\sqrt{x}\right)^2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Example:
Exercise:

Problem: Solve: $x-3\sqrt{x}+2=0$.

Solution:

The $\sqrt{x}$ in the middle term, is squared in the first term $\left(\sqrt{x}\right)^2=x$. If we let $u=\sqrt{x}$ and substitute, our trinomial will be in $ax^2+bx+c=$

0 form.

	$x-3\sqrt{x}+2=0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2-3\sqrt{x}+2=0$
Let $u=\sqrt{x}$ and substitute.	$u^2-3u+2=0$
Solve by factoring.	$(u-2)(u-1)=0$ $u-2=0,\quad u-1=0$ $u=2,\quad u=1$
Replace u with $\sqrt{x}$.	$\sqrt{x}=2,\quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4,\quad x=1$
Check:	

$$\begin{array}{rr} x=4 & x=1 \\ x-3\sqrt{x}+2=0 & x-3\sqrt{x}+2=0 \\ 4-3\sqrt{4}+2\stackrel{?}{=}0 & 1-3\sqrt{1}+2\stackrel{?}{=}0 \\ 4-6+2\stackrel{?}{=}0 & 1-3+2\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

Note:
Exercise:

Problem: Solve: $x - 7\sqrt{x} + 12 = 0$.

Solution:

$x = 9, x = 16$

Note:
Exercise:

Problem: Solve: $x - 6\sqrt{x} + 8 = 0$.

Solution:

$x = 4, x = 16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Example:
Exercise:

Problem: Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Solution:

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$
Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$ and substitute.	$u^2 - 2u - 24 = 0$
Solve by factoring.	$(u - 6)(u + 4) = 0$ $u - 6 = 0, \quad u + 4 = 0$ $u = 6, \quad u = -4$

Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}} = 6, \qquad x^{\frac{1}{3}} = -4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3 = (6)^3, \qquad \left(x^{\frac{1}{3}}\right)^3 = (-4)^3$ $x = 216, \qquad x = -64$
Check: $x = 216$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(216)^{\frac{2}{3}} - 2(216)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $36 - 12 - 24 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x = -64$ $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ $(-64)^{\frac{2}{3}} - 2(-64)^{\frac{1}{3}} - 24 \stackrel{?}{=} 0$ $16 + 8 - 24 \stackrel{?}{=} 0$ $0 = 0 \checkmark$	

Note:
Exercise:

Problem: Solve: $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0.$

Solution:

$x = -8, x = 343$

Note:
Exercise:

Problem: Solve: $x^{\frac{1}{2}} + 8x^{\frac{1}{4}} + 15 = 0$.

Solution:

$x = 81, x = 625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

Example:
Exercise:

Problem: Solve: $3x^{-2} - 7x^{-1} + 2 = 0$.

Solution:

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2 = x^{-2}$. If we let $u = x^{-1}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

		$3x^{-2} - 7x^{-1} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.		$3(x^{-1})^2 - 7(x^{-1}) + 2 = 0$

Let $u = x^{-1}$ and substitute.		$3u^2 - 7u + 2 = 0$
Solve by factoring.		$(3u - 1)(u - 2) = 0$
		$3u - 1 = 0, \quad u - 2 = 0$
		$u = \frac{1}{3}, \quad u = 2$
Replace u with x^{-1}.		$x^{-1} = \frac{1}{3}, \quad x^{-1} = 2$
Solve for x by taking the reciprocal since $x^{-1} = \frac{1}{x}$.		$x = 3, \quad x = \frac{1}{2}$
Check: $\begin{array}{rr} x = 3 & x = \frac{1}{2} \\ 3x^{-2} - 7x^{-1} + 2 = 0 & 3x^{-2} - 7x^{-1} + 2 = 0 \\ 3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0 & 3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0 \\ 3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0 & 3(4) - 7(2) + 2 \stackrel{?}{=} 0 \\ \frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0 & 12 - 14 + 2 \stackrel{?}{=} 0 \\ 0 = 0 ✓ & 0 = 0 ✓ \end{array}$		

Note:
Exercise:

Problem: Solve: $8x^{-2} - 10x^{-1} + 3 = 0$.

Solution:

$x = \frac{4}{3} x = 2$

Note:
Exercise:

Problem: Solve: $6x^{-2} - 23x^{-1} + 20 = 0$.

Solution:

$x = \frac{2}{5}, x = \frac{3}{4}$

Note:
Access this online resource for additional instruction and practice with solving quadratic equations.

- Solving Equations in Quadratic Form

Key Concepts

- How to solve equations in quadratic form.

  Identify a substitution that will put the equation in quadratic form.

Rewrite the equation with the substitution to put it in quadratic form.
Solve the quadratic equation for u.
Substitute the original variable back into the results, using the substitution.
Solve for the original variable.
Check the solutions.

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

Exercise:

Problem: $x^4 - 7x^2 + 12 = 0$

Solution:

$x = \pm\sqrt{3}, x = \pm 2$

Exercise:

Problem: $x^4 - 9x^2 + 18 = 0$

Exercise:

Problem: $x^4 - 13x^2 - 30 = 0$

Solution:

$x = \pm\sqrt{15}, x = \pm\sqrt{2}i$

Exercise:

Problem: $x^4 + 5x^2 - 36 = 0$

Exercise:

Problem: $2x^4 - 5x^2 + 3 = 0$

Solution:

$x = \pm 1, x = \frac{\pm\sqrt{6}}{2}$

Exercise:

Problem: $4x^4 - 5x^2 + 1 = 0$

Exercise:

Problem: $2x^4 - 7x^2 + 3 = 0$

Solution:

$x = \pm\sqrt{3}, x = \pm\frac{\sqrt{2}}{2}$

Exercise:

Problem: $3x^4 - 14x^2 + 8 = 0$

Exercise:

Problem: $(x-3)^2 - 5(x-3) - 36 = 0$

Solution:

$x = -1, x = 12$

Exercise:

Problem: $(x+2)^2 - 3(x+2) - 54 = 0$

Exercise:

Problem: $(3y+2)^2+(3y+2)-6=0$

Solution:

$x=-\frac{5}{3}, x=0$

Exercise:

Problem: $(5y-1)^2+3\left(5y-1\right)-28=0$

Exercise:

Problem: $\left(x^2+1\right)^2-5\left(x^2+1\right)+4=0$

Solution:

$x=0, x=\pm\sqrt{3}$

Exercise:

Problem: $\left(x^2-4\right)^2-4\left(x^2-4\right)+3=0$

Exercise:

Problem: $2\left(x^2-5\right)^2-5\left(x^2-5\right)+2=0$

Solution:

$x=\pm\frac{11}{2}, x=\pm\frac{\sqrt{22}}{2}$

Exercise:

Problem: $2\left(x^2-5\right)^2-7\left(x^2-5\right)+6=0$

Exercise:

Problem: $x - \sqrt{x} - 20 = 0$

Solution:

$x = 25$

Exercise:

Problem: $x - 8\sqrt{x} + 15 = 0$

Exercise:

Problem: $x + 6\sqrt{x} - 16 = 0$

Solution:

$x = 4$

Exercise:

Problem: $x + 4\sqrt{x} - 21 = 0$

Exercise:

Problem: $6x + \sqrt{x} - 2 = 0$

Solution:

$x = \frac{1}{4}$

Exercise:

Problem: $6x + \sqrt{x} - 1 = 0$

Exercise:

Problem: $10x - 17\sqrt{x} + 3 = 0$

Solution:

$x = \frac{1}{25}$

Exercise:

Problem: $12x + 5\sqrt{x} - 3 = 0$

Exercise:

Problem: $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

Solution:

$x = -1, x = -512$

Exercise:

Problem: $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

Exercise:

Problem: $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

Solution:

$x = 8, x = -216$

Exercise:

Problem: $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

Exercise:

Problem: $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

Solution:

$x = \frac{27}{8}, x = -\frac{64}{27}$

Exercise:

Problem: $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

Exercise:

Problem: $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

Solution:

$x = 27, x = 64{,}000$

Exercise:

Problem: $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

Exercise:

Problem: $x + 8x^{\frac{1}{2}} + 7 = 0$

Solution:

$x = 1, x = 49$

Exercise:

Problem: $2x - 7x^{\frac{1}{2}} = 15$

Exercise:

Problem: $6x^{-2} + 13x^{-1} + 5 = 0$

Solution:

$x = -2, x = -\frac{3}{5}$

Exercise:

Problem: $15x^{-2} - 26x^{-1} + 8 = 0$

Exercise:

Problem: $8x^{-2} - 2x^{-1} - 3 = 0$

Solution:

$x = -2, x = \frac{4}{3}$

Exercise:

Problem: $15x^{-2} - 4x^{-1} - 4 = 0$

Writing Exercises

Exercise:

Problem: Explain how to recognize an equation in quadratic form.

Solution:

Answers will vary.

Exercise:

Problem:

Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Solve Applications of Quadratic Equations
By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Note:
Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is −100. Find the numbers.
   If you missed this problem, review [link].
2. Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$.
   If you missed this problem, review [link].
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review [link].

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Note:
Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Note:
Use a Problem-Solving Strategy.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Solve the equation using algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one *n*, then the next one is *n* + 2. The next one would be *n* + 2 + 2 or *n* + 4. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

Equation:

	Consecutive even integers		**Consecutive odd integers**
	$64, 66, 68$		$77, 79, 81$
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example:

Exercise:

Problem: The product of two consecutive odd integers is 195. Find the integers.

Solution:

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive odd integers.

Step 3. Name what we are looking for. Let $n =$ the first odd integer.
$n+2 =$ the next odd integer

Step 4. Translate into an equation. State the problem in one sentence. "The product of two consecutive odd integers is 195."

The product of the first odd integer and the second odd integer is 195.

Translate into an equation. $n(n+2) = 195$

Step 5. Solve the equation. Distribute. $n^2 + 2n = 195$

Write the equation in standard form. $n^2 + 2n - 195 = 0$

Factor. $(n+15)(n-13) = 0$

Use the Zero Product Property. $n+15=0 \quad n-13=0$

Solve each equation. $n=-15, \quad n=13$

There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.

First odd integer $n=13$	First odd integer $n=-15$
next odd integer $n+2$	next odd integer $n+2$
$13+2$	$-15+2$
15	-13

Step 6. Check the answer.
Do these pairs work?
Are they consecutive odd integers?

$13, 15$ yes
$-13, -15$ yes

Is their product 195?

$13 \cdot 15 = 195$ yes
$-13(-15) = 195$ yes

Step 7. Answer the question. Two consecutive odd integers whose product is 195 are 13, 15 and -13, -15.

Note:
Exercise:

Problem: The product of two consecutive odd integers is 99. Find the integers.

Solution:

The two consecutive odd integers whose product is 99 are 9, 11, and −9, −11

Note:

Exercise:

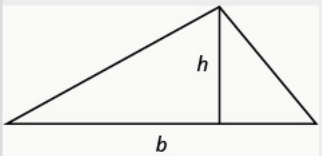
Problem: The product of two consecutive even integers is 168. Find the integers.

Solution:

The two consecutive even integers whose product is 128 are 12, 14 and −12, −14.

We will use the formula for the area of a triangle to solve the next example.

Note:
Area of a Triangle
For a triangle with base, b, and height, h, the area, A, is given by the formula $A=\frac{1}{2}bh$.



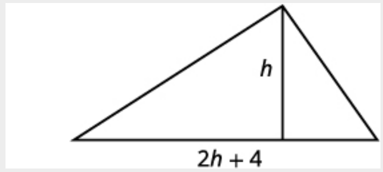
Recall that when we solve geometric applications, it is helpful to draw the figure.

Example:
Exercise:

Problem:

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution:

Step 1. Read the problem. Draw a picture.		
Step 2. Identify what we are looking for.		We are looking for the base and height.

Step 3. Name what we are looking for.		Let h = the height of the triangle. $2h + 4$ = the base of the triangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a triangle.		$A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.		$120 = \frac{1}{2}(2h + 4)h$
Distribute.		$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.		$h^2 + 2h - 120 = 0$
Factor.		$(h - 10)(h + 12) = 0$
Use the Zero Product Property.		$h - 10 = 0 \quad h + 12 = 0$
Simplify.		$h = 10, \qquad \cancel{h = -12}$
Since h is the height of a window, a value of $h = -12$ does not make sense.		
The height of the triangle $h = 10$.		
The base of the triangle $2h + 4$. $2 \cdot 10 + 4$ 24		
Step 6. Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.		
Step 7. Answer the question.		The height of the triangular window is 10 feet and the base is 24 feet.

Note:
Exercise:

Problem:

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Solution:

The height of the triangle is 12 inches and the base is 76 inches.

Note:
Exercise:

Problem:

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

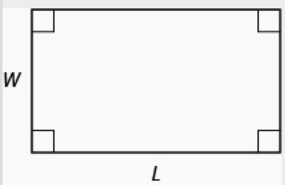
Solution:

The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Note:
Area of a Rectangle
For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.

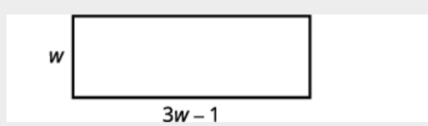


Example:
Exercise:

Problem:

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:

Step 1. **Read** the problem. Draw a picture.	
Step 2. **Identify** what we are looking for.	We are looking for the length and width.

Step 3. **Name** what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. **Solve** the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$ax^2 + bx + c = 0$ $3w^2 - w - 150 = 0$
Identify the a, b, c values.	$a = 3,\ b = -1,\ c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $w = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6}, \quad w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad \cancel{w \approx -6.9}$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6. **Check** the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Step 7. **Answer** the question.	The width of the rectangle is

	approximately 7.2 feet and the length is approximately 20.6 feet.

Note:
Exercise:

Problem:

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Solution:

The length of the garden is approximately 18 feet and the width 11 feet.

Note:
Exercise:

Problem:

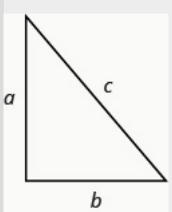
A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot.?

Solution:

The length of the tablecloth is approximatel 11.8 feet and the width 6.8 feet.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Note:
Pythagorean Theorem
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.

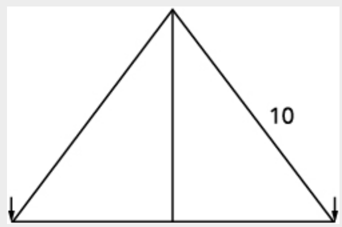
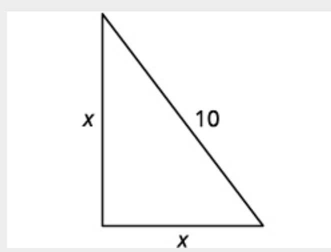


Example:
Exercise:

Problem:

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them. 
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*. Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$

Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}, \quad \cancel{x = -5\sqrt{2}}$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

Note:
Exercise:

Problem:

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Solution:

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

Note:
Exercise:

Problem:

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Solution:

The distance between the opposite corners is approximately 7.2 feet.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Note:
Projectile motion
The height in feet, h , of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula
Equation:

$$h = -16t^2 + v_0 t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

Example:
Exercise:

Problem:

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation. Use the formula.	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\begin{array}{cccc} ax^2 & + bx & + c & = 0 \\ 16t^2 & - 130t & + 260 & = 0 \end{array}$
Identify the values of a, b, c.	$a = 16, b = -130, c = 260$
Write the Quadratic Formula.	$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	

	$$t=\frac{-(-130)\pm\sqrt{(-130)^2-4\cdot 16\cdot(260)}}{2\cdot 16}$$
Simplify.	$$t=\frac{130\pm\sqrt{16{,}900-16{,}640}}{32}$$ $$t=\frac{130\pm\sqrt{260}}{32}$$
Rewrite to show two solutions.	$$t=\frac{130+\sqrt{260}}{32},\quad t=\frac{130-\sqrt{260}}{32}$$
Approximate the answer with a calculator.	$t\approx 4.6$ seconds, $t\approx 3.6$ seconds
Step 6. Check the answer. The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Note:
Exercise:

Problem:

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h=-16t^2+v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Solution:

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Note:
Exercise:

Problem:

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h=-16t^2+v_0t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Solution:

The ball will reach 48 feet on its way up after approximately .6 second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

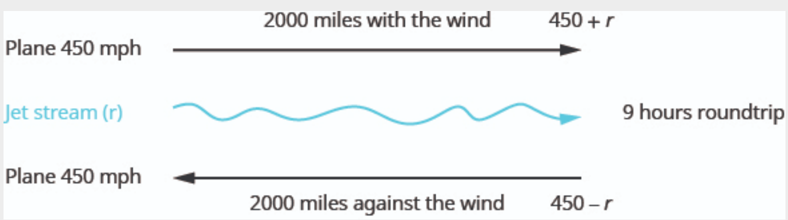
Example:
Exercise:

Problem:

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution:

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Write in the rates. Write in the distances.	

Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

Type	Rate	$\cdot$ Time	$=$ Distance
Headwind	$450 - r$	$\frac{2000}{450 - r}$	2000
Tailwind	$450 + r$	$\frac{2000}{450 + r}$	2000
		9	

We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450 - r)(450 + r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450 - r)(450 +$
Simplify.	$2000(450 + r) + 2000(450 - r) = 9(450 - r)(45$
Factor the 2,000.	$2000(450 + r + 450 - r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ $50 = r$ The speed of t
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500$ mph $\quad \frac{2000}{500} = 4$ hours Headwind $450 - 50 = 400$ mph $\quad \frac{2000}{400} = 5$ hours The times add to 9 hours, so it checks.	
	The speed of the jet stream was 50 mph.

Note:
Exercise:

Problem:

MaryAnne just returned from a visit with her grandchildren back east . The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Solution:

The speed of the jet stream was 100 mph.

Note:
Exercise:

Problem:

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Solution:

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We’ll use a similar scenario now.

Example:
Exercise:

Problem:

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution:

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

<table>
<tr><td>Let $x =$ the number of hours for Press #2 to complete the job.
Enter the hours per job for Press #1, Press #2, and when they work together.</td><td><table><tr><th></th><th>Number of hours needed to complete the job.</th><th>Part of job completed/hour</th></tr><tr><td>Press #1</td><td>$x + 12$</td><td>$\frac{1}{x+12}$</td></tr><tr><td>Press #2</td><td>x</td><td>$\frac{1}{x}$</td></tr><tr><td>Together</td><td>8</td><td>$\frac{1}{8}$</td></tr></table></td></tr>
<tr><td>The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.
Translate to an equation.</td><td>$\underbrace{\text{Press \#1} + \text{Press \#2} = \text{Together}}^{\text{Work completed by}}$
$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$</td></tr>
<tr><td>Solve.</td><td>$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$</td></tr>
</table>

Multiply by the LCD, $8x\left(x+12\right)$.	$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x+8(x+12)=x(x+12)$ $8x+8x+96=x^2+12x$ $0=x^2-4x-96$
Solve.	$0=(x-12)(x+8)$ $x-12=0,\ x+8=0$ $x=12,\ \cancel{x=-8}$ hours
Since the idea of negative hours does not make sense, we use the value $x=12$.	$12+12$ 12 24 hours 12 hours
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

Note:
Exercise:

Problem:

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Solution:

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Note:
Exercise:

Problem:

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Solution:

The red hose take 6 hours and the green hose take 3 hours alone.

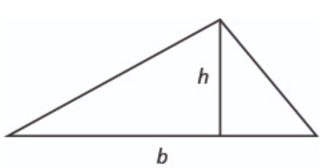
Note:
Access these online resources for additional instruction and practice with solving applications modeled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

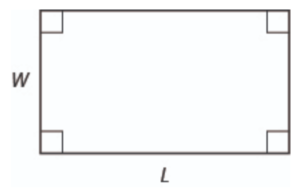
Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.

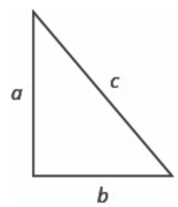
  Readthe problem. Make sure all the words and ideas are understood.
  Identifywhat we are looking for.
  Namewhat we are looking for. Choose a variable to represent that quantity.
  Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  Solvethe equation using good algebra techniques.
  Checkthe answer in the problem and make sure it makes sense.
  Answerthe question with a complete sentence.
- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.



- Area of a Rectangle
  - For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.



- Pythagorean Theorem
    - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.



- Projectile motion
    - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0 t$.

Practice Makes Pefect

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.
Exercise:

Problem: The product of two consecutive odd numbers is 255. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 255 are 15 and 17, and −15 and −17.

Exercise:

Problem: The product of two consecutive even numbers is 360. Find the numbers.

Exercise:

Problem: The product of two consecutive even numbers is 624. Find the numbers.

Solution:

The first and second consecutive odd numbers are 24 and 26, and −26 and −24.

Exercise:

Problem: The product of two consecutive odd numbers is 1,023. Find the numbers.

Exercise:

Problem: The product of two consecutive odd numbers is 483. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 483 are 21 and 23, and −21 and −23.

Exercise:

Problem: The product of two consecutive even numbers is 528. Find the numbers.

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

Exercise:

Problem:

A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.

Solution:

The width of the triangle is 5 inches and the height is 18 inches.

Exercise:

Problem:

The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.

Exercise:

Problem:

The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?

Solution:

The base is 24 feet and the height of the triangle is 10 feet.

Exercise:

Problem:

A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?

Exercise:

Problem:

The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.

Solution:

The length of the driveway is 15.0 feet and the width is 3.3 feet.

Exercise:

Problem:

A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

Exercise:

Problem:

A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.

Solution:

The length of table is 8 feet and the width is 3 feet.

Exercise:

Problem:

The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?

Exercise:

Problem:

The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

Solution:

The length of the legs of the right triangle are 3.2 and 9.6 cm.

Exercise:

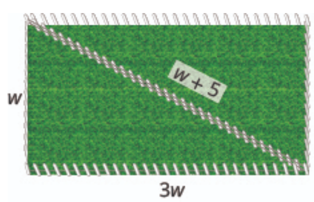
Problem:

The hypotenuse of a right triangle is 10 cm long. One of the triangle’s legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.

Exercise:

Problem:

A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.



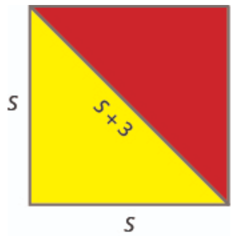
Solution:

The length of the diagonal fencing is 7.3 yards.

Exercise:

Problem:

Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



Exercise:

Problem:

Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

Solution:

The ladder will reach 24.5 feet on the side of the house.

Exercise:

Problem:

John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

Exercise:

Problem:

A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0t$ to determine when the height of the firework rocket will be 1200 feet.

Solution:

The arrow will reach 400 feet on its way up in 2.8 seconds and on the way down in 11 seconds.

Exercise:

Problem:

An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0t$, to determine when height of the arrow will be 400 feet.

Exercise:

Problem:

A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

Solution:

The bullet will take 70 seconds to hit the ground.

Exercise:

Problem:

A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0 t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

Exercise:

Problem:

The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

Solution:

The speed of the wind was 49 mph.

Exercise:

Problem:

The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

Exercise:

Problem:

Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

Solution:

The speed of the current was 4.3 mph.

Exercise:

Problem:

Rick paddled up the river, spent the night camping, and and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

Exercise:

Problem:

Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

Solution:

The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.

Exercise:

Problem:

Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintainence individually?

Exercise:

Problem:

It takes two hours for two machines to manufacture 10,000 parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture 10,000 parts alone?

Solution:

Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

Exercise:

Problem:

Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the hot tub?

Writing Exercises

Exercise:

Problem: Make up a problem involving the product of two consecutive odd integers.

ⓐ Start by choosing two consecutive odd integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Solution:

Answers will vary.

Exercise:

Problem: Make up a problem involving the product of two consecutive even integers.

ⓐ Start by choosing two consecutive even integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Graph Quadratic Functions Using Properties
By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Note:
Before you get started, take this readiness quiz.

1. Graph the function $f\left(x\right)=x^2$ by plotting points.
   If you missed this problem, review [link].
2. Solve: $2x^2+3x-2=0$.
   If you missed this problem, review [link].
3. Evaluate $-\frac{b}{2a}$ when *a* = 3 and *b* = −6.
   If you missed this problem, review [link].

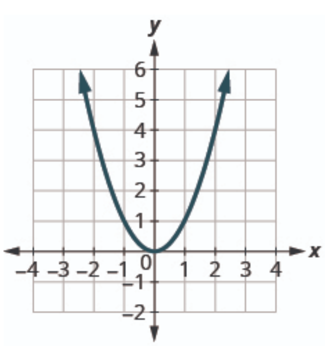
Recognize the Graph of a Quadratic Function

Previously we very briefly looked at the function $f(x)=x^2$, which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form $f\left(x\right)=ax^2+bx+c$ if $a\neq 0$. We call this kind of function a quadratic function.

Note:
Quadratic Function
A **quadratic function**, where a, b, and c are real numbers and $a\neq 0$, is a function of the form
Equation:

$$f\left(x\right)=ax^2+bx+c$$

We graphed the quadratic function $f(x)=x^2$ by plotting points.



x	$f(x)=x^2$	$(x, f(x))$
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

Every quadratic function has a graph that looks like this. We call this figure a **parabola**.

Let's practice graphing a parabola by plotting a few points.

Example:
Exercise:

Problem: Graph $f\left(x\right)=x^{2}-1$.

Solution:

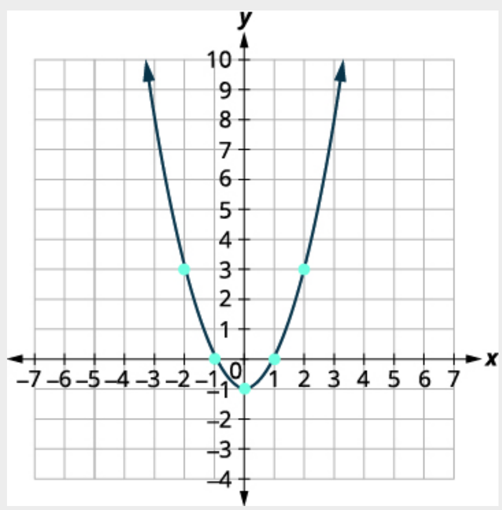
We will graph the function by plotting points.

Choose integer values for *x*, substitute them into the equation and simplify to find $f\left(x\right)$.

Record the values of the ordered pairs in the chart.

$f(x)=x^2-1$	
x	$f(x)$
0	–1
1	0
–1	0
2	3
–2	3

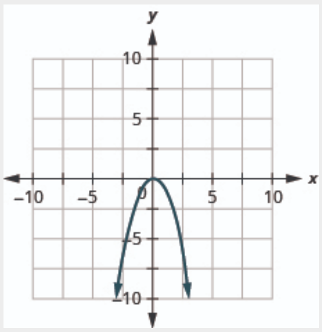
Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f\left(x\right)=x^{2}-1$.



Note:
Exercise:

Problem: Graph $f(x) = -x^2$..

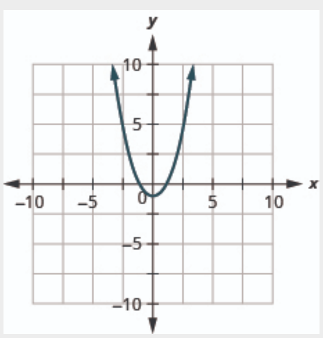
Solution:



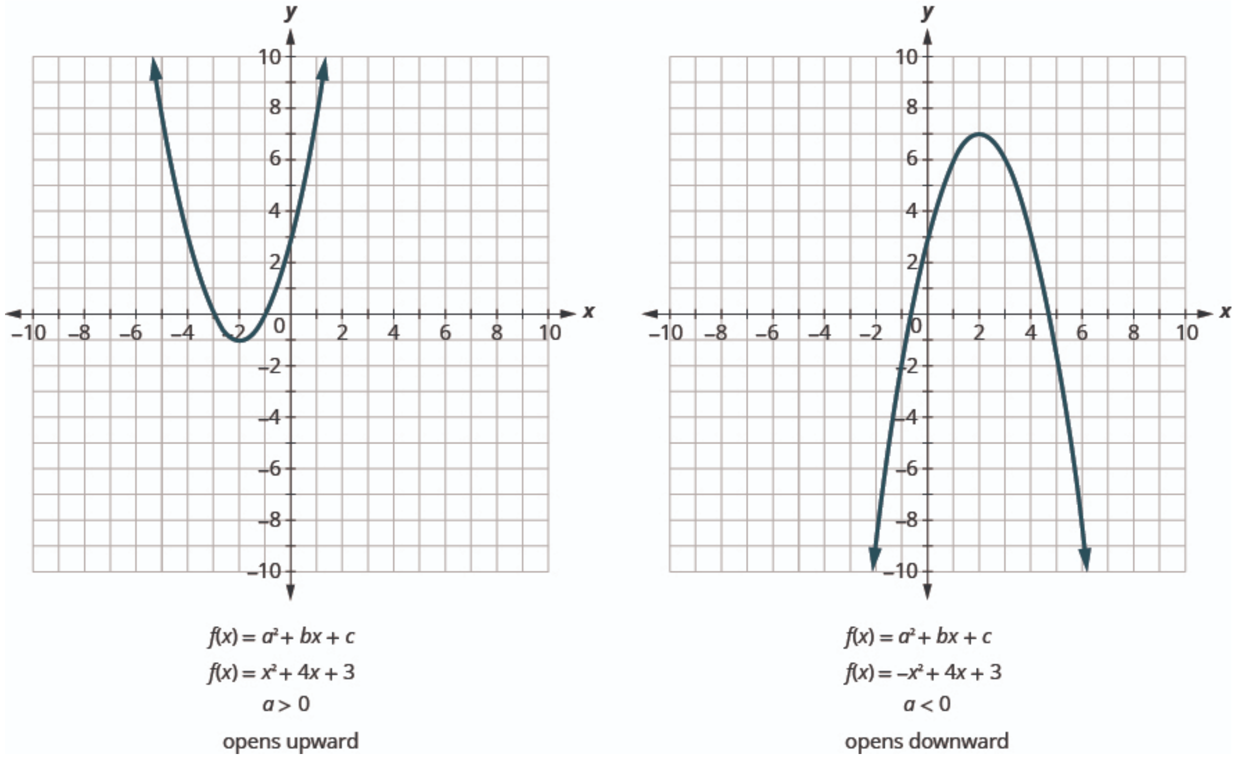
Note:
Exercise:

Problem: Graph $f(x) = x^2 + 1$.

Solution:

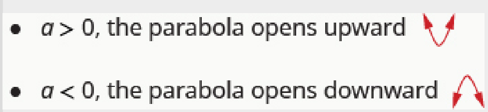


All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See [link].



Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in [link]). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Note:
Parabola Orientation
For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if



Example:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x^2 + 2x - 4$ ⓑ $f(x) = 6x^2 + 7x - 9$.

Solution:

ⓐ

Find the value of “*a*”.	$f(x) = ax^2 + bx + c$ $f(x) = -3x^2 + 2x - 4$ $a = -3$
	Since the “*a*” is negative, the parabola will open downward.

ⓑ

Find the value of “*a*”.	$f(x) = ax^2 + bx + c$ $f(x) = 6x^2 + 7x - 9$ $a = 6$
	Since the “*a*” is positive, the parabola will open upward.

Note:
Exercise:

Problem: Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = 2x^2 + 5x - 2$ ⓑ $f(x) = -3x^2 - 4x + 7$.

Solution:

ⓐ up; ⓑ down

Note:
Exercise:

Problem: Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x) = -2x^2 - 2x - 3$ ⓑ $f(x) = 5x^2 - 2x - 1$.

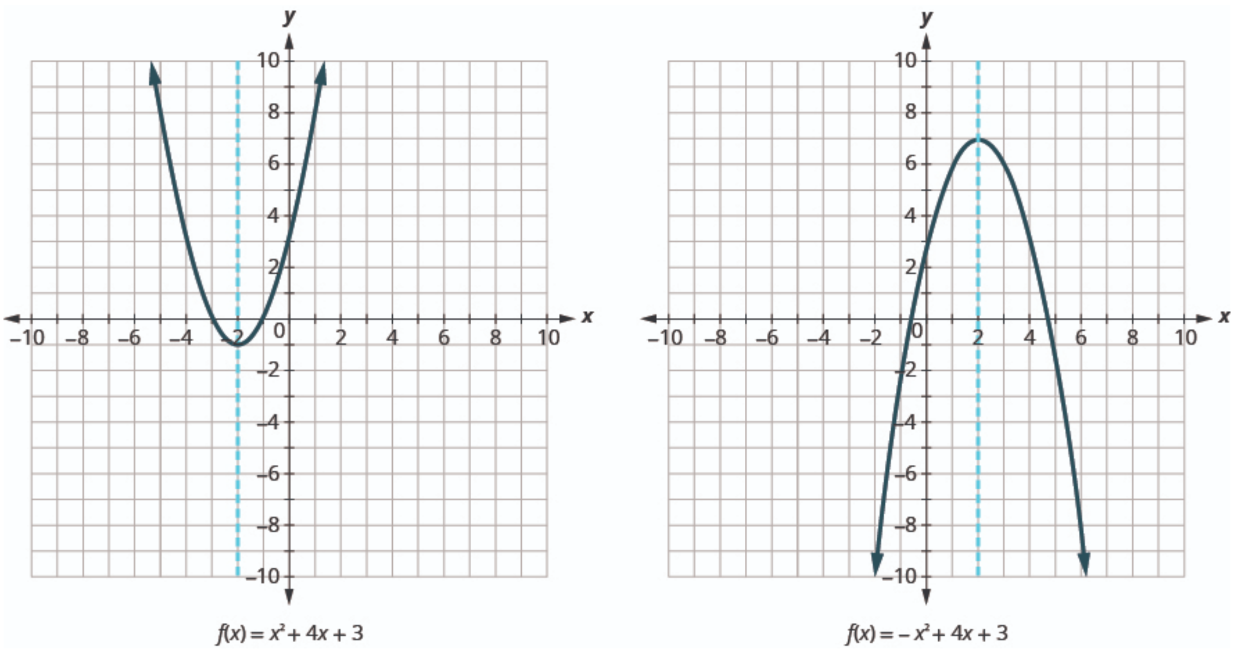
Solution:

ⓐ down; ⓑ up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at [link]. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See [link].



The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$	$f(x) = ax^2 + bx + c$
$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its

x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$	$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_\_)$	vertex is $(2, \_\_\_)$
$f(x) = x^2 + 4x + 3$ $f(x) = (-2)^2 + 4(-2) + 3$ $f(x) = -1$	$f(x) = -x^2 + 4x + 3$ $f(x) = -(2)^2 + 4(2) + 3$ $f(x) = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Note:
Axis of Symmetry and Vertex of a Parabola
The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example:
Exercise:

Problem: For the graph of $f(x) = 3x^2 - 6x + 2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Solution:

ⓐ

	$f(x) = ax^2 + bx + c$ $f(x) = 3x^2 - 6x + 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values of a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$

Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

ⓑ

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its *x*-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$
The result is the *y*-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

Note:
Exercise:

Problem: For the graph of $f(x) = 2x^2 - 8x + 1$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Solution:

ⓐ $x = 2$; ⓑ $(2, -7)$

Note:
Exercise:

Problem: For the graph of $f(x) = 2x^2 - 4x - 3$ find:

ⓐ the axis of symmetry ⓑ the vertex.

Solution:

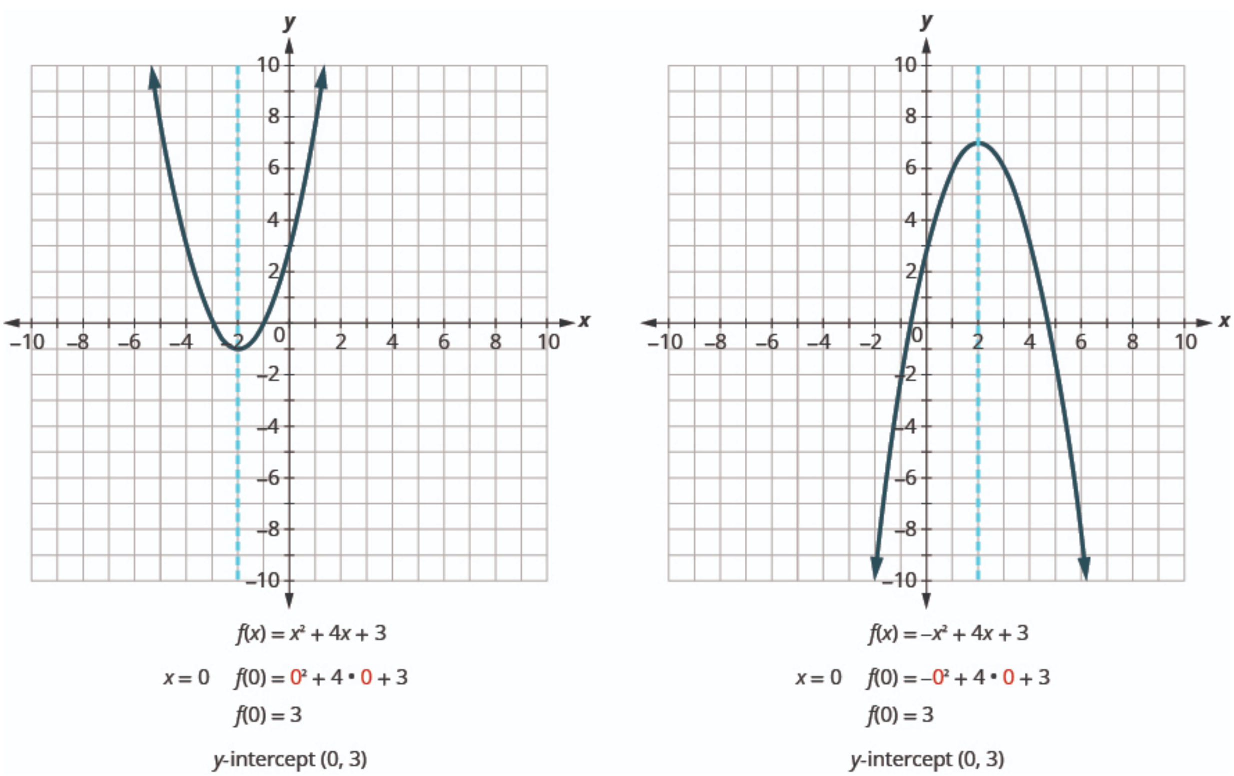
ⓐ $x = 1$; ⓑ $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute *x* = 0 into the function.

Let’s find the *y*-intercepts of the two parabolas shown in [link].



An *x*-intercept results when the value of $f(x)$ is zero. To find an *x*-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for *x*.
Equation:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ 0 &= ax^2 + bx + c \end{aligned}$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the *x*-intercepts of the two parabolas we looked at. First we will find the *x*-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

	$f(x) = x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x+1)(x+3)$
Use the Zero Product Property.	$x + 1 = 0 \quad x + 3 = 0$
Solve.	$x = -1 \qquad x = -3$
	The *x*-intercepts are $(-1, 0)$ and $(-3, 0)$.

Now we will find the *x*-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

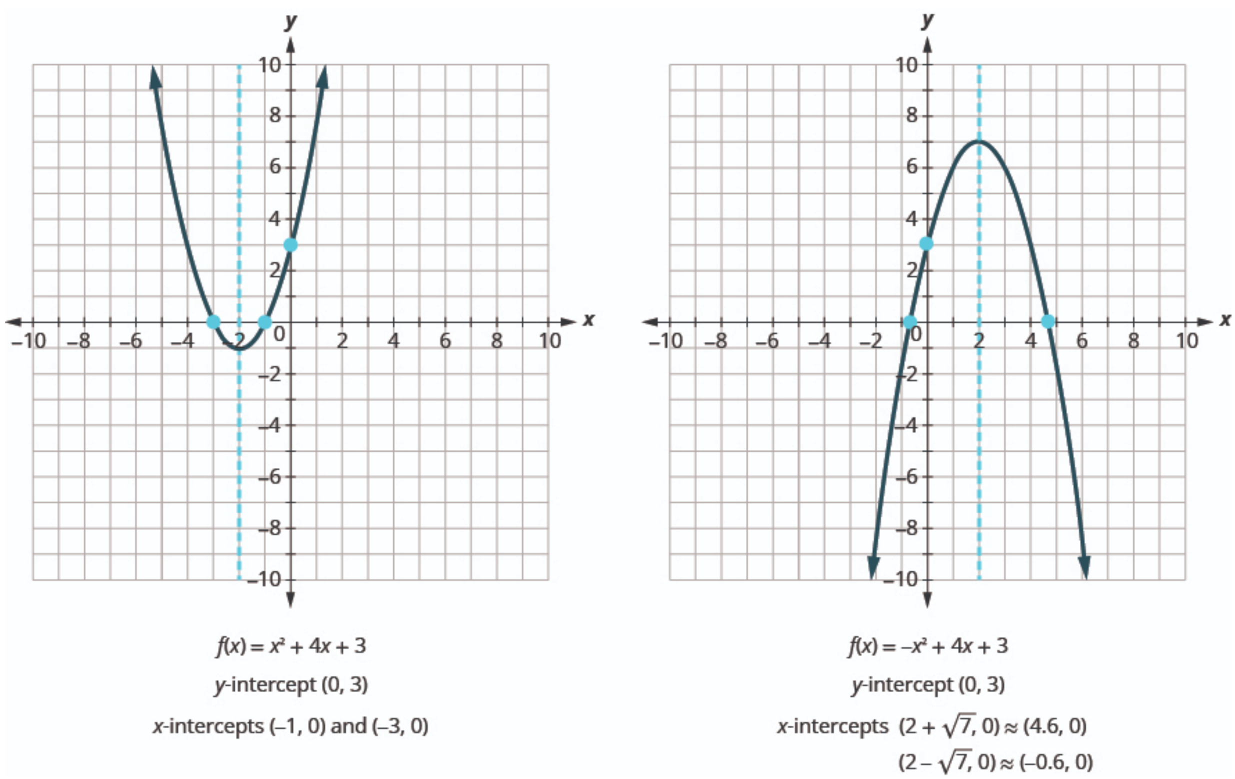
	$f(x) = -x^2 + 4x + 3$
Let $f(x) = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = -1, b = 4, c = 3$	$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$
Simplify.	$x = \frac{-4 \pm \sqrt{28}}{-2}$

	$x = \frac{-4 \pm 2\sqrt{7}}{-2}$
	$x = \frac{-2(2 \pm \sqrt{7})}{-2}$
	$x = 2 \pm \sqrt{7}$
	The x-intercepts are $\left(2+\sqrt{7}, 0\right)$ and $\left(2-\sqrt{7}, 0\right)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,
Equation:

$$\left(2+\sqrt{7}, 0\right) \approx (4.6, 0) \qquad \left(2-\sqrt{7}, 0\right) \approx (-0.6, 0)$$

Do these results agree with our graphs? See [link].



Note:
Find the Intercepts of a Parabola
To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:
Equation:

y**-intercept**	x**-intercepts**
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

Example:
Exercise:

Problem: Find the intercepts of the parabola whose function is $f(x) = x^2 - 2x - 8$.

Solution:

To find the ***y***-intercept, let $x = 0$ and solve for $f(x)$.		$f(x) = x^2 - 2x - 8$
		$f(0) = 0^2 - 2 \cdot 0 - 8$
		$f(0) = -8$
	When $x = 0$, then $f(0) = -8$. The *y*-intercept is the point $(0, -8)$.	
To find the ***x***-intercept, let $f(x) = 0$ and solve for x.		$f(x) = x^2 - 2x - 8$
		$0 = x^2 - 2x - 8$
Solve by factoring.		$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$	
	$4 = x \quad -2 = x$	

	When $f(x) = 0$, then $x = 4$ or $x = -2$. The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Note:
Exercise:

Problem: Find the intercepts of the parabola whose function is $f(x) = x^2 + 2x - 8$.

Solution:

y-intercept: $(0, -8)$ *x*-intercepts $(-4, 0), (2, 0)$

Note:
Exercise:

Problem: Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 12$.

Solution:

y-intercept: $(0, -12)$ *x*-intercepts $(-2, 0), (6, 0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for *x* and the results were the solutions to the equation.

We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The *x*-intercepts of the parabolas occur where $f(x) = 0$.

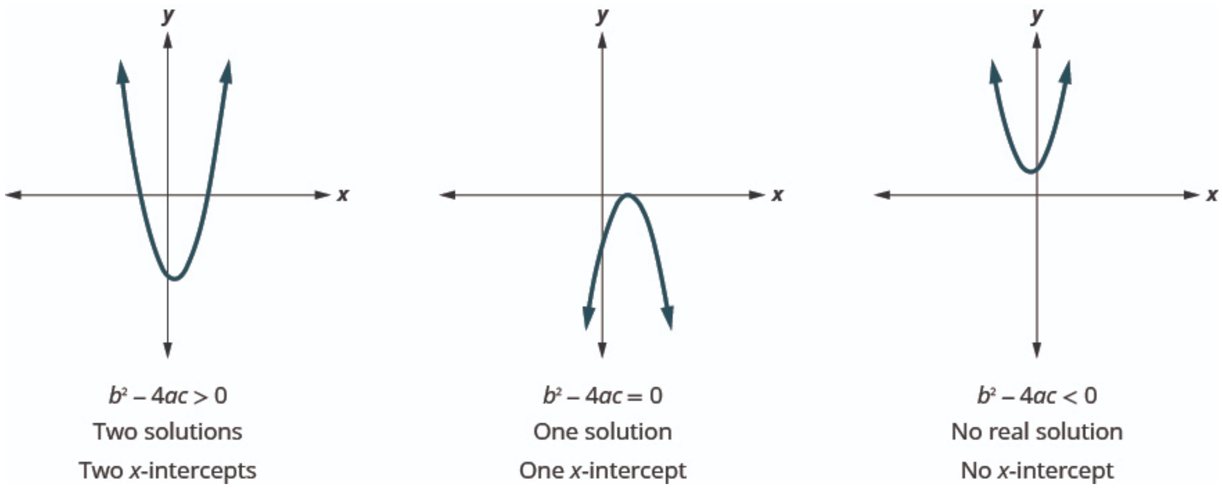
For example:
Equation:

$$\begin{array}{ccc} \textbf{Quadratic equation} & & \textbf{Quadratic function} \\ \begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x - 5 = 0 \quad x + 3 &= 0 \\ x = 5 \qquad x &= -3 \end{aligned} & \text{Let } f(x) = 0. & \begin{aligned} f(x) &= x^2 - 2x - 15 \\ 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \\ x - 5 &= 0 \quad x + 3 = 0 \\ x &= 5 \qquad x = -3 \end{aligned} \\ & & (5, 0) \text{ and } (-3, 0) \\ & & x\text{-intercepts} \end{array}$$

The solutions of the quadratic function are the *x* values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax^2 + bx + c = 0$. Now we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example:
Exercise:

Problem: Find the intercepts of the parabola for the function $f(x) = 5x^2 + x + 4$.

Solution:

	$f(x) = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for $f(x)$.	$f(0) = 5 \cdot 0^2 + 0 + 4$
	$f(0) = 4$
	When $x = 0$, then $f(0) = 4$. The *y*-intercept is the point $(0, 4)$.

To find the *x*-intercept, let $f(x) = 0$ and solve for x.	$f(x) = 5x^2 + x + 4$
	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions which is also the number of *x*-intercepts.	
$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79	
	Since the value of the discriminant is negative, there is no real solution to the equation. There are no *x*-intercepts.

Note:
Exercise:

Problem: Find the intercepts of the parabola whose function is $f(x) = 3x^2 + 4x + 4$.

Solution:

y-intercept: $(0, 4)$ no *x*-intercept

Note:
Exercise:

Problem: Find the intercepts of the parabola whose function is $f(x) = x^2 - 4x - 5$.

Solution:

y-intercept: $(0, -5)$ *x*-intercepts $(-1, 0), (5, 0)$

Graph Quadratic Functions Using Properties

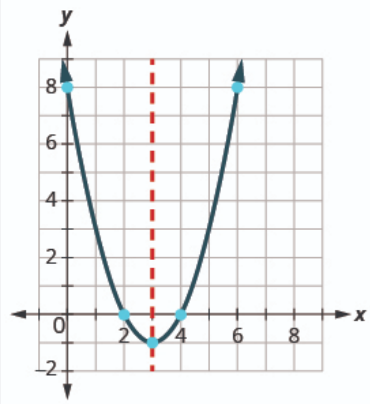
Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

Example:
How to Graph a Quadratic Function Using Properties
Exercise:

Problem: Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Solution:

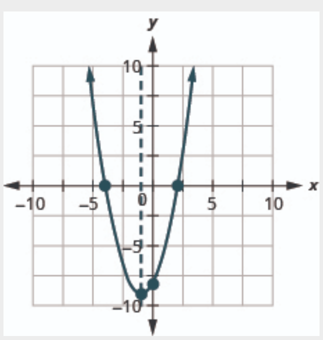
Step 1. Determine whether the parabola opens upward or downward.	Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward. 	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**
Step 2. Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 3. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, –1).**
Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).** Point symmetric to y-intercept: **The point is (6, 8).**

Step 5. Find the *x*-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	*x*-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The *x*-intercepts are (2, 0) and (4, 0).**
Step 6. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

Note:
Exercise:

Problem: Graph $f(x) = x^2 + 2x - 8$ by using its properties.

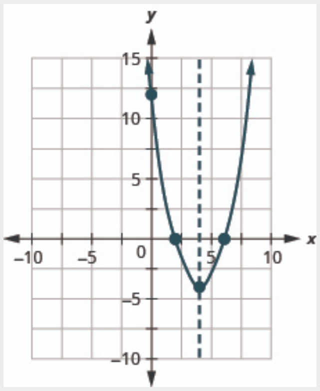
Solution:



Note:
Exercise:

Problem: Graph $f(x) = x^2 - 8x + 12$ by using its properties.

Solution:



We list the steps to take in order to graph a quadratic function here.

Note:
To graph a quadratic function using properties.

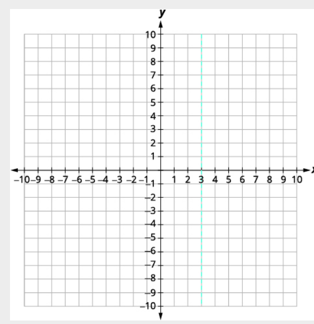
Determine whether the parabola opens upward or downward.
Find the equation of the axis of symmetry.
Find the vertex.
Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
Find the x-intercepts. Find additional points if needed.
Graph the parabola.

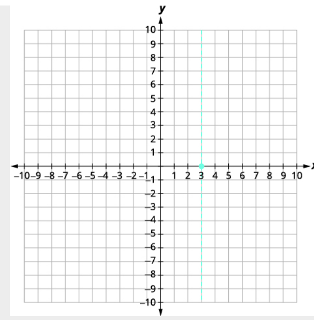
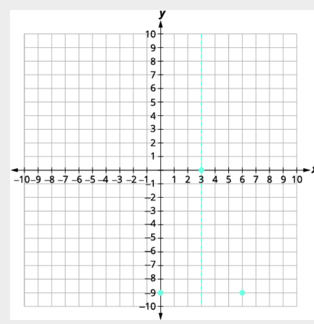
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

Example:
Exercise:

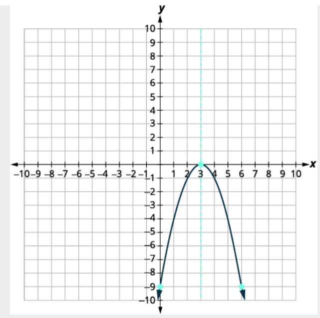
Problem: Graph $f(x) = x^2 + 6x - 9$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find $f(3)$.	$f(x) = -x^2 + 6x - 9$
	$f(3) = -3^2 + 6 \cdot 3 - 9$
	$f(3) = -9 + 18 - 9$
	$f(3) = 0$
	The vertex is $(3, 0)$.

	
The *y*-intercept occurs when $x = 0$. Find $f(0)$.	$f(x) = -x^2 + 6x - 9$
Substitute $x = 0$.	$f(0) = -0^2 + 6 \cdot 0 - 9$
Simplify.	$f(0) = -9$
	The *y*-intercept is $(0, -9)$.
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the *y*-intercept is $(6, -9)$
The *x*-intercept occurs when $f(x) = 0$.	$f(x) = -x^2 + 6x - 9$
Find $f(x) = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for *x*.	$0 = 3$

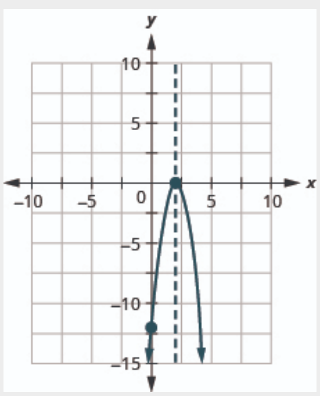
Connect the points to graph the parabola.



Note:
Exercise:

Problem: Graph $f(x) = 3x^2 + 12x - 12$ by using its properties.

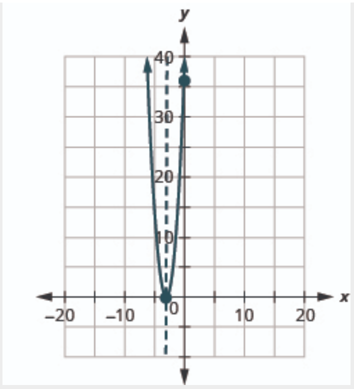
Solution:



Note:
Exercise:

Problem: Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.

Solution:



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

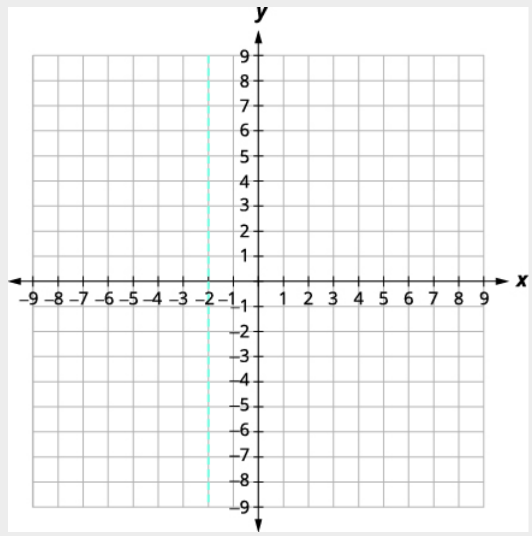
How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

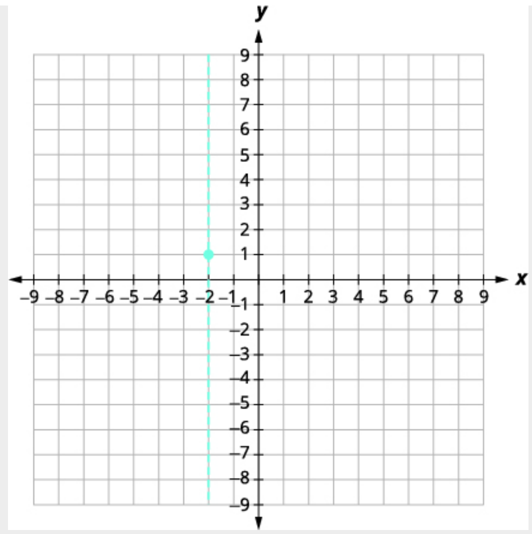
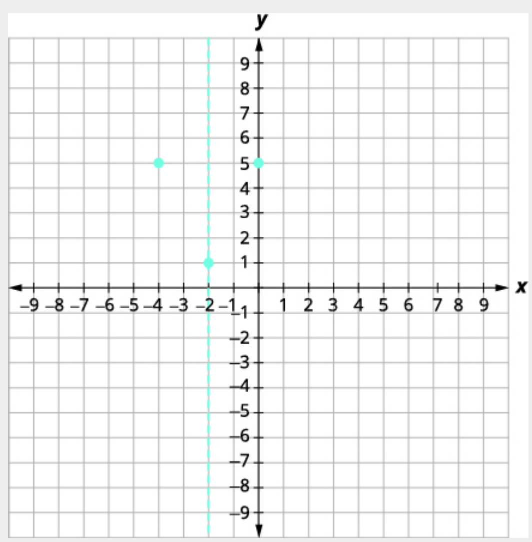
Example:
Exercise:

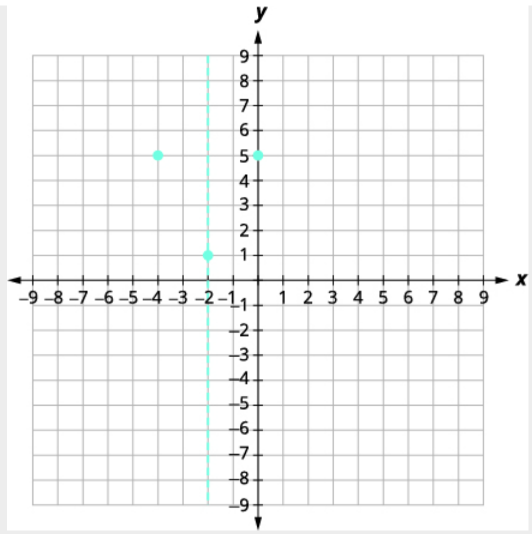
Problem: Graph $f(x) = x^2 + 4x + 5$ by using its properties.

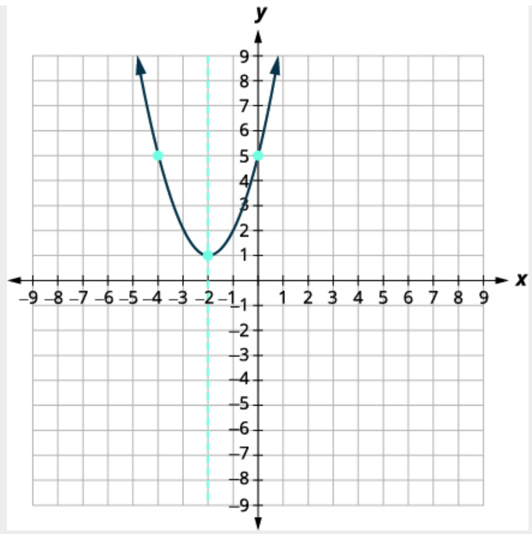
Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = x^2 + 4x + 5$
Since a is 1, the parabola opens upward.	
	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{4}{(2)1}$

	$x = -2$
	The equation of the axis of symmetry is $x = -2$.
	
The vertex is on the line $x = -2$.	
Find $f(x)$ when $x = -2$.	$f(x) = x^2 + 4x + 5$
	$f(-2) = (-2)^2 + 4(-2) + 5$
	$f(-2) = 4 - 8 + 5$
	$f(-2) = 1$
	The vertex is $(-2, 1)$.

	
The *y*-intercept occurs when $x = 0$.	$f(x) = x^2 + 4x - 5$
Find $f(0)$.	$f(0) = 5$
Simplify.	$f(0) = 5$
	The *y*-intercept is $(0, 5)$.
The point $(-4, 5)$ is two units to the left of the line of symmetry. The point two units to the right of the line of symmetry is $(0, 5)$.	
	Point symmetric to the *y*-intercept is $(-4, 5)$.
The *x*-intercept occurs when $f(x) = 0$.	

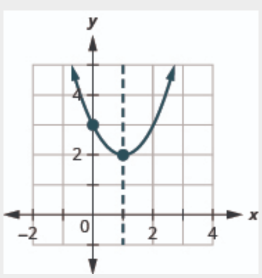
	
Find $f(x) = 0$.	$0 = x^2 + 4x + 5$
Test the discriminant.	
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no *x*-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.	



Note:
Exercise:

Problem: Graph $f(x) = x^2 - 2x + 3$ by using its properties.

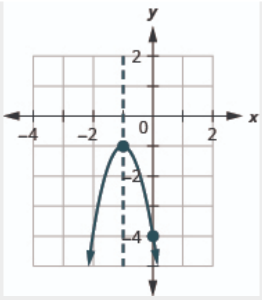
Solution:



Note:
Exercise:

Problem: Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.

Solution:



Finding the *y*-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the *x*-intercepts.

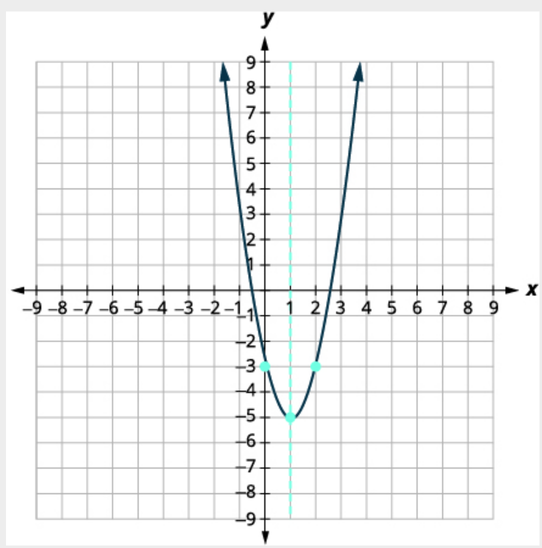
Example:
Exercise:

Problem: Graph $f(x) = 2x^2 - 4x - 3$ by using its properties.

Solution:

	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
Since *a* is 2, the parabola opens upward. 	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{-4}{2 \cdot 2}$
	$x = 1$
	The equation of the axis of symmetry is $x = 1$.

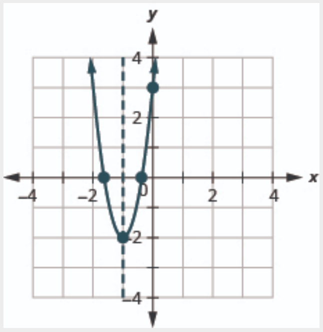
The vertex is on the line $x = 1$.	$f(x) = 2x^2 - 4x - 3$
Find $f(1)$.	$f(x) = 2(1)^2 - 4(1) - 3$
	$f(1) = 2 - 4 - 3$
	$f(1) = -5$
	The vertex is $(1, -5)$.
The y-intercept occurs when $x = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(0)$.	$f(0) = 2(0)^2 - 4(0) - 3$
Simplify.	$f(0) = -3$
	The y-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry.	**Point symmetric to the y-intercept is $(2, -3)$**
The point one unit to the right of the line of symmetry is $(2, -3)$.	
The x-intercept occurs when $y = 0$.	$f(x) = 2x^2 - 4x - 3$
Find $f(x) = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of a, b, and c.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$
Simplify.	$x = \frac{-4 \pm \sqrt{16 + 24}}{4}$

Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, \quad x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, \quad x \approx -0.6$
	The approximate values of the *x*-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

Note:
Exercise:

Problem: Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

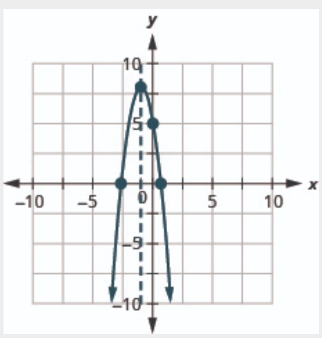
Solution:



Note:
Exercise:

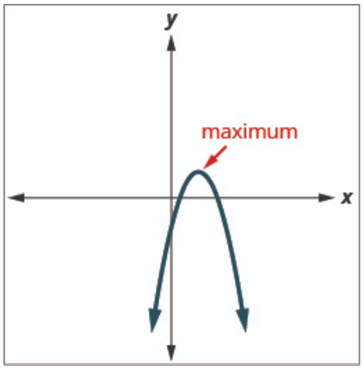
Problem: Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.

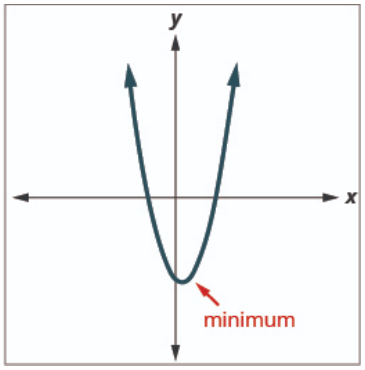
Solution:



Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The y-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See [link].





Note:
Minimum or Maximum Values of a Quadratic Function
The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

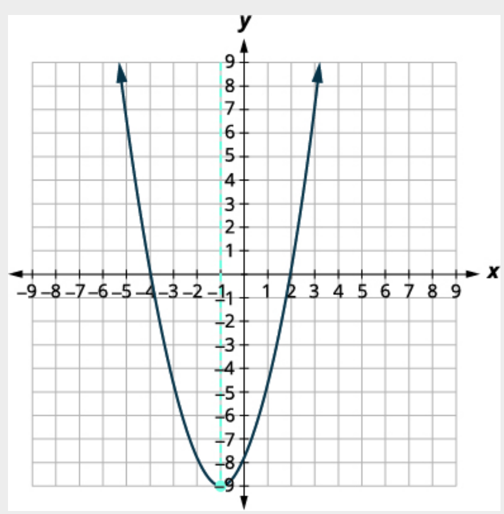
- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Example:
Exercise:

Problem: Find the minimum or maximum value of the quadratic function $f(x) = x^2 + 2x - 8$.

Solution:

	$f(x) = x^2 + 2x - 8$
Since *a* is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2 \times 1}$
	$x = -1$

	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$f(x) = x^2 + 2x - 8$
Find $f(-1)$.	$f(-1) = (-1)^2 + 2(-1) - 8$
	$f(-1) = 1 - 2 - 8$
	$f(-1) = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	
	
Show the graph to verify the result.	

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic function $f\left(x\right)=x^2-8x+12.$

Solution:

The minimum value of the quadratic function is −4 and it occurs when $x = 4$.

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic function $f\left(x\right)=-4x^2+16x-11.$

Solution:

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula
Equation:

$$h\left(t\right)=-16t^2+v_0t+h_0$$

to calculate the height in feet, h , of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

Example:
Exercise:

Problem:

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height? ⓑ Find the maximum height of the volleyball.

Solution:

$$h\left(t\right)=-16t^2+176t+4$$

Since a is negative, the parabola opens downward.
The quadratic function has a maximum.

ⓐ

Find the equation of the axis of symmetry.

$t = -\frac{b}{2a}$

$t = -\frac{176}{2(-16)}$

$t = 5.5$

The equation of the axis of symmetry is $t = 5.5$.

The vertex is on the line $t = 5.5$.

The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find $h(5.5)$.

$h(t) = -16t^2 + 176t + 4$

$h(t) = -16(5.5)^2 + 176(5.5) + 4$

Use a calculator to simplify.

$h(t) = 488$

The vertex is $(5.5, 488)$.

Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

Note:
Exercise:

Problem: Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Solution:

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

Note:
Exercise:

Problem:

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

Solution:

It will 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Note:
Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- Quadratic Functions: Axis of Symmetry and Vertex
- Finding x- and y-intercepts of a Quadratic Function
- Graphing Quadratic Functions
- Solve Maxiumum or Minimum Applications
- Quadratic Applications: Minimum and Maximum

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function $f\left(x\right)=a{x}^{2}+bx+c$, if
    - *a* > 0, the parabola opens upward.
    - *a* < 0, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f\left(x\right)=a{x}^{2}+bx+c$ is a parabola where:
  - the axis of symmetry is the vertical line $x=-\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
  - the *y*-coordinate of the vertex is found by substituting $x=-\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose function is $f\left(x\right)=a{x}^{2}+bx+c$:
    Equation:

$$\begin{array}{cc}\textbf{y-intercept} & \textbf{x-intercepts}\\ \text{Let } x=0 \text{ and solve for } f\left(x\right). & \text{Let } f\left(x\right)=0 \text{ and solve for } x.\end{array}$$

- How to graph a quadratic function using properties.

  Determine whether the parabola opens upward or downward.
  Find the equation of the axis of symmetry.
  Find the vertex.
  Find the*y*-intercept. Find the point symmetric to the*y*-intercept across the axis of symmetry.
  Find the*x*-intercepts. Find additional points if needed.
  Graph the parabola.

- Minimum or Maximum Values of a Quadratic Equation
  - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

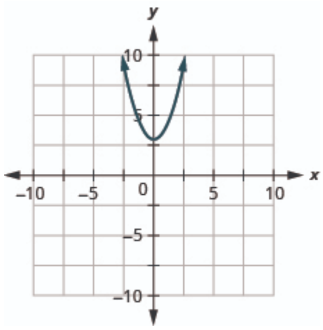
Practice Makes Perfect

Recognize the Graph of a Quadratic Function

In the following exercises, graph the functions by plotting points.
Exercise:

Problem: $f(x) = x^2 + 3$

Solution:



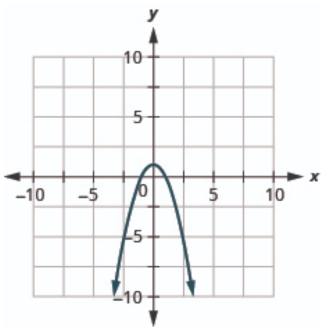
Exercise:

Problem: $f\left(x\right) = x^2 - 3$

Exercise:

Problem: $y = -x^2 + 1$

Solution:



Exercise:

Problem: $f\left(x\right) = -x^2 - 1$

For each of the following exercises, determine if the parabola opens up or down.
Exercise:

Problem: ⓐ $f(x) = -2x^2 - 6x - 7$
ⓑ $f(x) = 6x^2 + 2x + 3$

Solution:

ⓐ down ⓑ up

Exercise:

Problem: ⓐ $f(x) = 4x^2 + x - 4$
ⓑ $f(x) = -9x^2 - 24x - 16$

Exercise:

Problem: ⓐ $f\left(x\right) = -3x^2 + 5x - 1$
ⓑ $f\left(x\right) = 2x^2 - 4x + 5$

Solution:

ⓐ down ⓑ up

Exercise:

Problem: ⓐ $f\left(x\right) = x^2 + 3x - 4$
ⓑ $f\left(x\right) = -4x^2 - 12x - 9$

Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.

Exercise:

Problem: $f\left(x\right) = x^2 + 8x - 1$

Solution:

ⓐ $x = -4$; ⓑ $(-4, -17)$

Exercise:

Problem: $f\left(x\right) = x^2 + 10x + 25$

Exercise:

Problem: $f\left(x\right) = -x^2 + 2x + 5$

Solution:

ⓐ $x = 1$; ⓑ $(1, 2)$

Exercise:

Problem: $f\left(x\right)=-2{x}^{2}-8x-3$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.
Exercise:

Problem: $f(x)={x}^{2}+7x+6$

Solution:

y-intercept: $(0,6)$; *x*-intercept $(-1,0),(-6,0)$

Exercise:

Problem: $f(x)={x}^{2}+10x-11$

Exercise:

Problem: $f\left(x\right)={x}^{2}+8x+12$

Solution:

y-intercept: $(0,12)$; *x*-intercept $(-2,0),(-6,0)$

Exercise:

Problem: $f\left(x\right)={x}^{2}+5x+6$

Exercise:

Problem: $f(x)=-{x}^{2}+8x-19$

Solution:

y-intercept: $(0,-19)$; *x*-intercept: none

Exercise:

Problem: $f\left(x\right)=-3{x}^{2}+x-1$

Exercise:

Problem: $f(x)={x}^{2}+6x+13$

Solution:

y-intercept: $(0,13)$; *x*-intercept: none

Exercise:

Problem: $f\left(x\right)={x}^{2}+8x+12$

Exercise:

Problem: $f(x) = 4x^2 - 20x + 25$

Solution:

y-intercept: $(0, -16)$; *x*-intercept $\left(\frac{5}{2}, 0\right)$

Exercise:

Problem: $f(x) = -x^2 - 14x - 49$

Exercise:

Problem: $f\left(x\right) = -x^2 - 6x - 9$

Solution:

y-intercept: $(0, 9)$; *x*-intercept $(-3, 0)$

Exercise:

Problem: $f\left(x\right) = 4x^2 + 4x + 1$

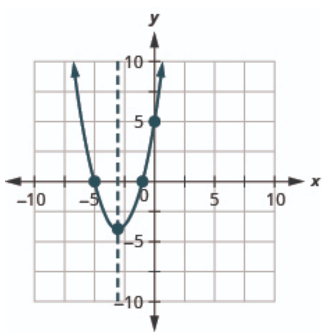
Graph Quadratic Functions Using Properties

In the following exercises, graph the function by using its properties.

Exercise:

Problem: $f(x) = x^2 + 6x + 5$

Solution:



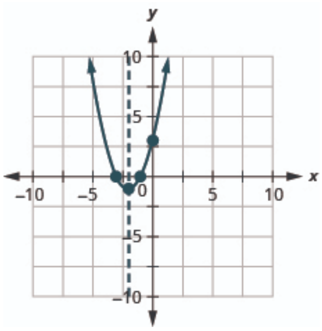
Exercise:

Problem: $f(x) = x^2 + 4x - 12$

Exercise:

Problem: $f(x) = x^2 + 4x + 3$

Solution:



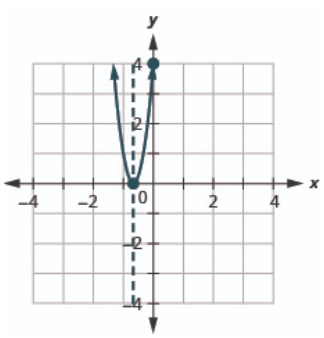
Exercise:

Problem: $f(x) = x^2 - 6x + 8$

Exercise:

Problem: $f(x) = 9x^2 + 12x + 4$

Solution:



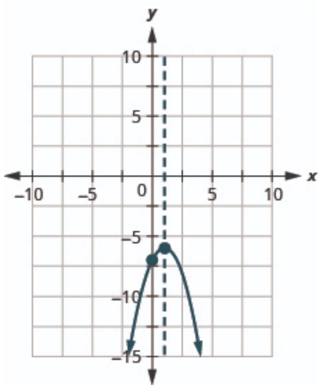
Exercise:

Problem: $f(x) = -x^2 + 8x - 16$

Exercise:

Problem: $f(x) = -x^2 + 2x - 7$

Solution:



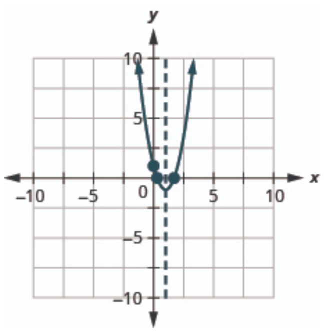
Exercise:

Problem: $f(x) = 5x^2 + 2$

Exercise:

Problem: $f(x) = 2x^2 - 4x + 1$

Solution:



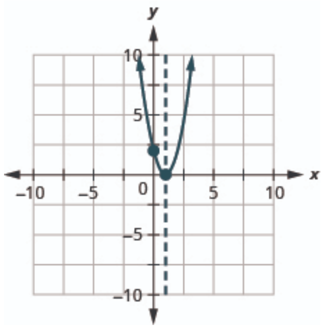
Exercise:

Problem: $f(x) = 3x^2 - 6x - 1$

Exercise:

Problem: $f(x) = 2x^2 - 4x + 2$

Solution:



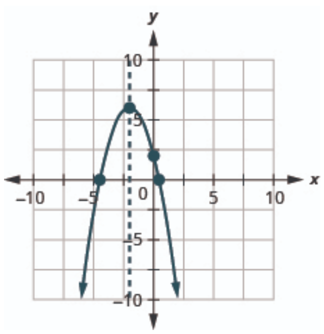
Exercise:

Problem: $f(x) = -4x^2 - 6x - 2$

Exercise:

Problem: $f(x) = -x^2 - 4x + 2$

Solution:



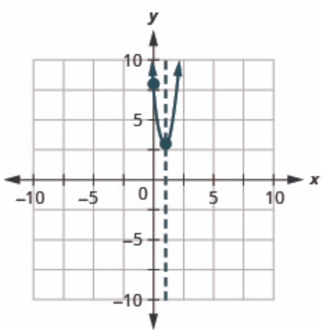
Exercise:

Problem: $f(x) = x^2 + 6x + 8$

Exercise:

Problem: $f(x) = 5x^2 - 10x + 8$

Solution:



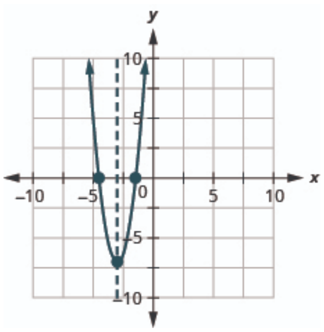
Exercise:

Problem: $f(x) = -16x^2 + 24x - 9$

Exercise:

Problem: $f(x) = 3x^2 + 18x + 20$

Solution:



Exercise:

Problem: $f(x) = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

Exercise:

Problem: $f\left(x\right) = 2x^2 + x - 1$

Solution:

The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

Exercise:

Problem: $y = -4x^2 + 12x - 5$

Exercise:

Problem: $y = x^2 - 6x + 15$

Solution:

The maximum value is 6 when *x* = 3.

Exercise:

Problem: $y = -x^2 + 4x - 5$

Exercise:

Problem: $y=-9x^2+16$

Solution:

The maximum value is 16 when $x = 0$.

Exercise:

Problem: $y=4x^2-49$

In the following exercises, solve. Round answers to the nearest tenth.

Exercise:

Problem:

An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

Solution:

In 5.3 sec the arrow will reach maximum height of 486 ft.

Exercise:

Problem:

A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

Exercise:

Problem:

A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

Solution:

In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

Exercise:

Problem:

A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maxiumum height, and then find the maximum height.

Exercise:

Problem:

A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Solution:

20 computers will give the maximum of $400 in receipts.

Exercise:

Problem:

A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Exercise:

Problem:

A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x^2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Solution:

He will be able to sell 35 pairs of boots at the maximum revenue of $1,225.

Exercise:

Problem:

A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x^2 + 8x$ to find the revenue received when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue, and then find the amount of the maximum revenue.

Exercise:

Problem:

A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A(x) = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

Solution:

The length of the side along the river of the corral is 120 feet and the maximum area is 7,200 square feet.

Exercise:

Problem:

A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

Exercise:

Problem:

A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

Solution:

The maximum area of the patio is 800 feet.

Exercise:

Problem:

A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x) = x(300 - 2x)$ to determine the maximum area of the fenced in yard.

Writing Exercise

Exercise:

Problem:

How do the graphs of the functions $f\left(x\right)=x^2$ and $f\left(x\right)=x^2-1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

Solution:

Answers will vary.

Exercise:

Problem: Explain the process of finding the vertex of a parabola.

Exercise:

Problem: Explain how to find the intercepts of a parabola.

Solution:

Answers will vary.

Exercise:

Problem: How can you use the discriminant when you are graphing a quadratic function?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

quadratic function
: A quadratic function, where a, b, and c are real numbers and $a \neq 0$, is a function of the form $f\left(x\right) = ax^2 + bx + c$.

Solve Quadratic Inequalities
By the end of this section, you will be able to:

- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Note:
Before you get started, take this readiness quiz.

1. Solve: $2x-3=0$.
   If you missed this problem, review [link].
2. Solve: $2y^2+y=15$.
   If you missed this problem, review [link].
3. Solve $\frac{1}{x^2+2x-8}>0$
   If you missed this problem, review [link].

We have learned how to solve linear inequalities and rational inequalities previously. Some of the techniques we used to solve them were the same and some were different.

We will now learn to solve inequalities that have a quadratic expression. We will use some of the techniques from solving linear and rational inequalities as well as quadratic equations.

We will solve quadratic inequalities two ways—both graphically and algebraically.

Solve Quadratic Inequalities Graphically

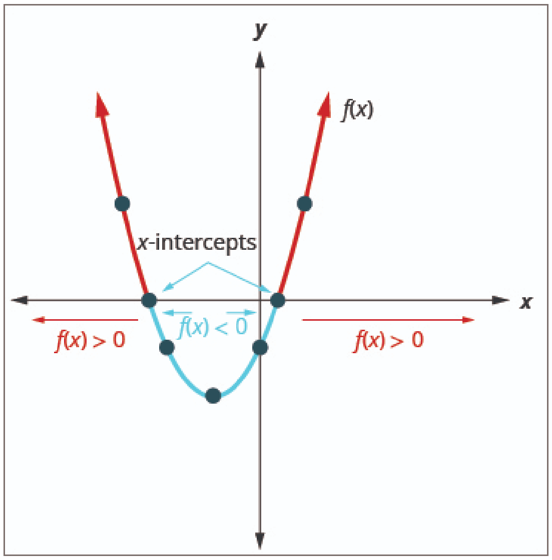
A quadratic equation is in standard form when written as $ax^2 + bx + c = 0$. If we replace the equal sign with an inequality sign, we have a **quadratic inequality** in standard form.

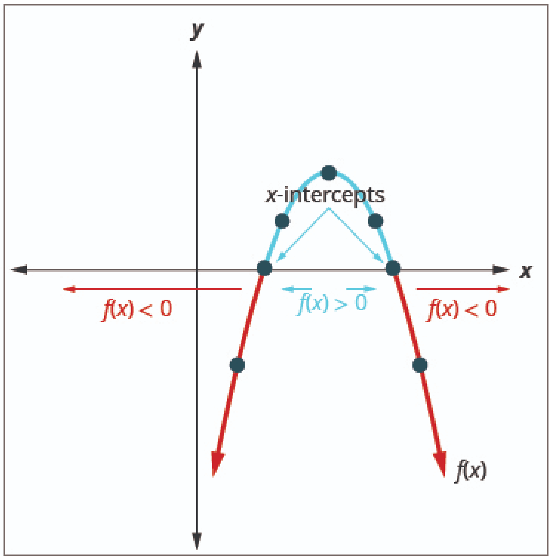
Note:
Quadratic Inequality
A **quadratic inequality** is an inequality that contains a quadratic expression.
The standard form of a quadratic inequality is written:
Equation:

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c \leq 0$$
$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c \geq 0$$

The graph of a quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola. When we ask when is $ax^2 + bx + c < 0$, we are asking when is $f(x) < 0$. We want to know when the parabola is below the x-axis.

When we ask when is $ax^2 + bx + c > 0$, we are asking when is $f(x) > 0$. We want to know when the parabola is above the y-axis.





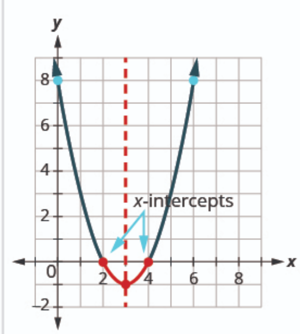
Example:
How to Solve a Quadratic Inequality Graphically
Exercise:

Problem:

Solve $x^2 - 6x + 8 < 0$ graphically. Write the solution in interval notation.

Solution:

Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - 6x + 8 < 0$

Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.	We will graph using the properties. Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward. 	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**
	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, −1).**
	We find $f(0)$	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept **The point is (6, 8).**
	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	x-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**
	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Step 3. Determine the solution from the graph.	$x^2 - 6x + 8 < 0$ The inequality asks for the values of x which make the function less than 0. Which values of x make the parabola below the x-axis. We do not include the values 2, 4 as the inequality is less than only.	The solution, in interval notation, is (2, 4).

Note:

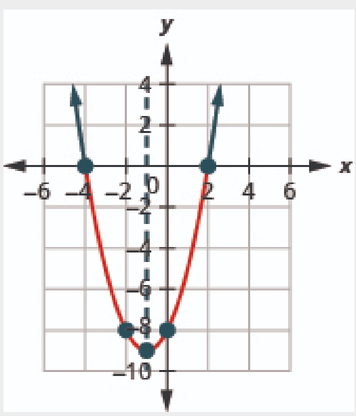
Exercise:

Problem:

ⓐ Solve $x^2 + 2x - 8 < 0$ graphically and ⓑ write the solution in interval notation.

Solution:

ⓐ



ⓑ $(-4, -2)$

Note:

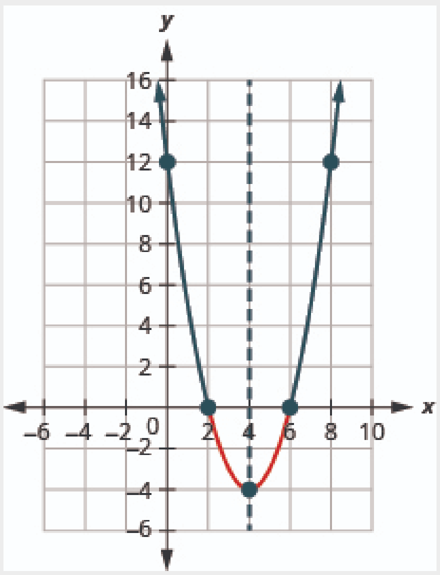
Exercise:

Problem:

ⓐ Solve $x^2 - 8x + 12 \geq 0$ graphically and ⓑ write the solution in interval notation.

Solution:

ⓐ



ⓑ $(-\infty, 2] \cup [6, \infty)$

We list the steps to take to solve a quadratic inequality graphically.

Note:
Solve a quadratic inequality graphically.

Write the quadratic inequality in standard form.
Graph the function $f(x) = ax^2 + bx + c$.

Determine the solution from the graph.

In the last example, the parabola opened upward and in the next example, it opens downward. In both cases, we are looking for the part of the parabola that is below the *x*-axis but note how the position of the parabola affects the solution.

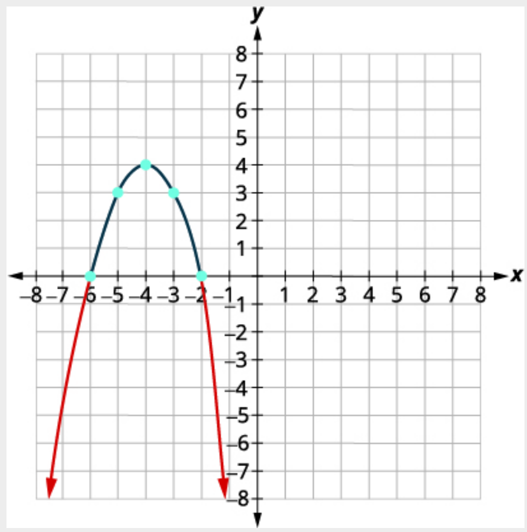
Example:
Exercise:

Problem:

Solve $-x^2-8x-12\le 0$ graphically. Write the solution in interval notation.

Solution:

The quadratic inequality in standard form.		$-x^2-8x-12\le 0$
Graph the function $f\left(x\right)=-x^2-8x-12$.		The parabola opens downward. 
Find the line of symmetry.		$x=-\frac{b}{2a}$ $x=-\frac{-8}{2(-1)}$ $x=-4$
Find the vertex.		$f(x)=-x^2-8x-12$

		$f(-4) = -(-4)^2 - 8(-4) - 12$ $f(-4) = -16 + 32 - 12$ $f(-4) = 4$ Vertex $(-4, 4)$
Find the *x*-intercepts. Let $f(x) = 0$.		$f(x) = -x^2 - 8x - 12$ $0 = -x^2 - 8x - 12$
Factor. Use the Zero Product Property.		$0 = -1(x + 6)(x + 2)$ $x = -6 \quad x = -2$
Graph the parabola.		*x*-intercepts $(-6, 0), (-2, 0)$ 
Determine the solution from the graph. We include the *x*-intercepts as the inequality is “less than or equal to.”		$(-\infty,\ -6] \cup [-2,\ \infty)$

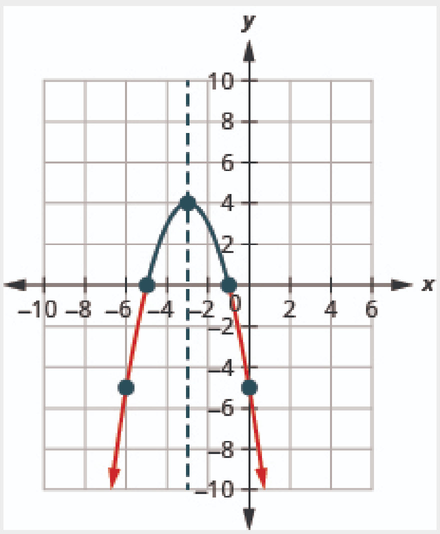
Note:
Exercise:

Problem:

ⓐ Solve $-x^2 - 6x - 5 > 0$ graphically and ⓑ write the solution in interval notation.

Solution:

ⓐ



ⓑ $(-1, 5)$

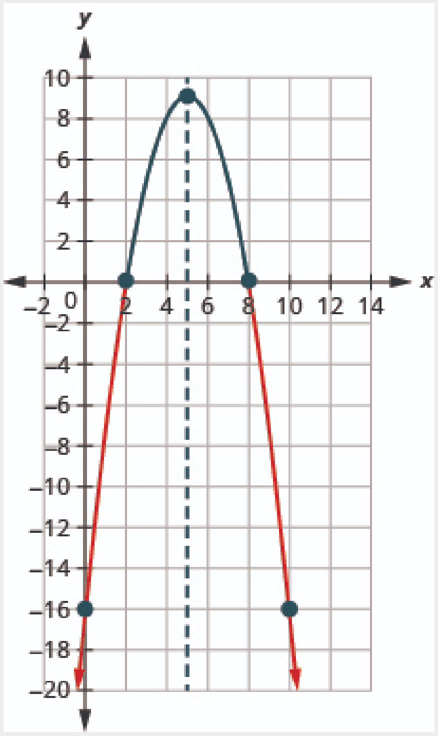
Note:
Exercise:

Problem:

ⓐ Solve $-x^2 + 10x - 16 \leq 0$ graphically and ⓑ write the solution in interval notation.

Solution:

ⓐ



ⓑ $(-\infty, 2] \cup [8, \infty)$

Solve Quadratic Inequalities Algebraically

The algebraic method we will use is very similar to the method we used to solve rational inequalities. We will find the critical points for the inequality, which will be the solutions to the related quadratic equation. Remember a polynomial expression can change signs only where the expression is zero.

We will use the critical points to divide the number line into intervals and then determine whether the quadratic expression willl be postive or negative in the interval. We then determine the solution for the inequality.

Example:

How To Solve Quadratic Inequalities Algebraically

Exercise:

Problem:

Solve $x^2 - x - 12 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Step 1. Write the quadratic inequality in standard form.	The inequality is in standard form	$x^2 - x - 12 \geq 0$
Step 2. Determine the critical points—the solutions to the related quadratic equation.	Change the inequality sign to an equal sign and then solve the equation.	$x^2 - x - 12 = 0$ $(x + 3)(x - 4) = 0$ $x + 3 = 0 \quad x - 4 = 0$ $x = -3 \quad x = 4$
Step 3. Use the critical points to divide the number line into intervals.	Use –3 and 4 to divide the number line into intervals	
Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.	Test: $x = -5$ $x = 0$ $x = 5$	$x^2 - x - 12$ / $x^2 - x - 12$ / $x^2 - x - 12$ $(-5)^2 - (-5) - 12$ / $0^2 - 0 - 12$ / $5^2 - 5 - 12$ 18 / -12 / 8 
Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.	$x^2 - x - 12 \geq 0$ The inequality is positive in the first and last intervals and equals 0 at the points –4, 3.	The solution, in interval notation, is $(-\infty, -3] \cup [4, \infty)$.

Note:

Exercise:

Problem:

Solve $x^2 + 2x - 8 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

$(-\infty, -4] \cup [2, \infty)$

Note:
Exercise:

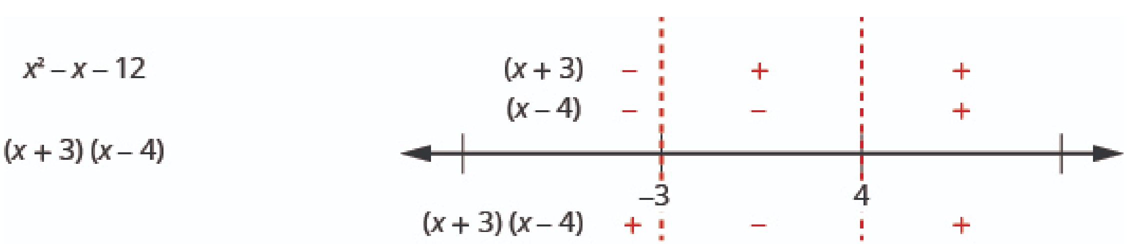
Problem:

Solve $x^2 - 2x - 15 \leq 0$ algebraically. Write the solution in interval notation.

Solution:

$[-3, 5]$

In this example, since the expression $x^2 - x - 12$ factors nicely, we can also find the sign in each interval much like we did when we solved rational inequalities. We find the sign of each of the factors, and then the sign of the product. Our number line would like this:



The result is the same as we found using the other method.

We summarize the steps here.

Note:
Solve a quadratic inequality algebraically.

Write the quadratic inequality in standard form.
Determine the critical points—the solutions to the related quadratic equation.
Use the critical points to divide the number line into intervals.
Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
Determine the intervals where the inequality is correct. Write the solution in interval notation.

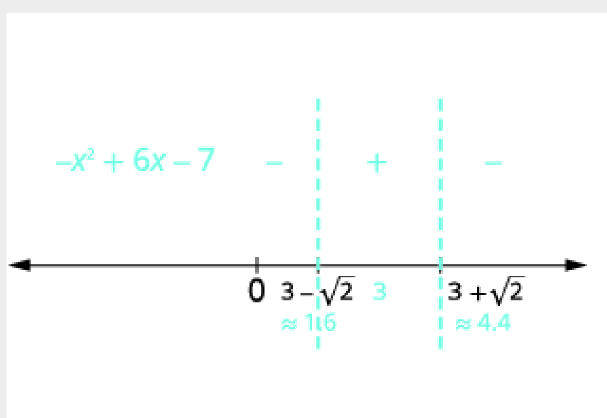
Example:
Exercise:

Problem:

Solve $x^2 + 6x - 7 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

Write the quadratic inequality in standard form.	$-x^2 + 6x - 7 \geq 0$
Multiply both sides of the inequality by -1. Remember to	$x^2 - 6x + 7 \leq 0$

reverse the inequality sign.	
Determine the critical points by solving the related quadratic equation.	$x^2 - 6x + 7 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (7)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{8}}{2}$
Simplify the radical.	$x = \frac{6 \pm 2\sqrt{2}}{2}$
Remove the common factor, 2.	$x = \frac{2\left(3 \pm \sqrt{2}\right)}{2}$ $x = 3 \pm \sqrt{2}$ $x = 3 + \sqrt{2}$ $\quad$ $x = 3 - \sqrt{2}$ $x \approx 1.6$ $\quad$ $x \approx 4.4$
Use the critical points to divide the number line into intervals. Test numbers from each interval in the original inequality.	

Determine the intervals where the inequality is correct. Write the solution in interval notation.	$-x^2+6x-7 \geq 0$ in the middle interval $\left[3-\sqrt{2},\ 3+\sqrt{2}\right]$

Note:
Exercise:

Problem:

Solve $-x^2+2x+1 \geq 0$ algebraically. Write the solution in interval notation.

Solution:

$\left[-1-\sqrt{2}, -1+\sqrt{2}\right]$

Note:
Exercise:

Problem:

Solve $-x^2+8x-14 < 0$ algebraically. Write the solution in interval notation.

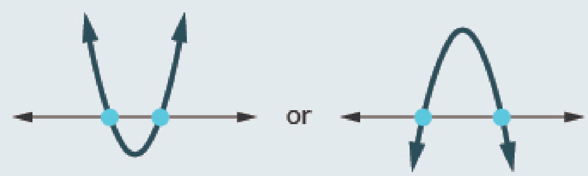
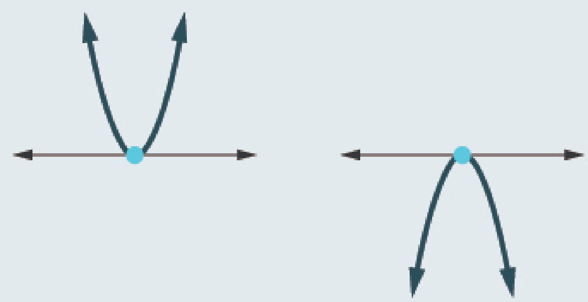
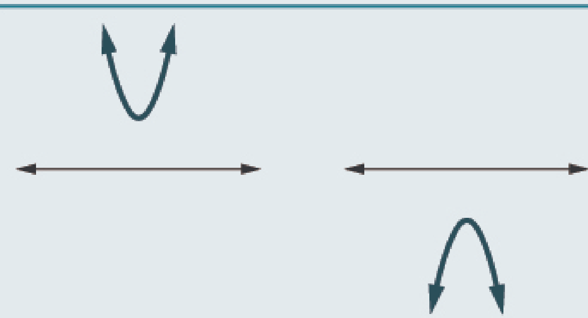
Solution:

$\left(-\infty, 4-\sqrt{2}\right) \cup \left(4+\sqrt{2}, \infty\right)$

The solutions of the quadratic inequalities in each of the previous examples, were either an interval or the union of two intervals. This resulted from the fact that, in each case we found two solutions to the corresponding quadratic equation $ax^2 + bx + c = 0$. These two solutions then gave us either the two x-intercepts for the graph or the two critical points to divide the number line into intervals.

This correlates to our previous discussion of the number and type of solutions to a quadratic equation using the discriminant.

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$.

Discriminant	Number/Type of solution	Typical Graph
$b^2 - 4ac > 0$	2 real solutions 2 x-intercepts on graph	
$b^2 - 4ac = 0$	1 real solution 1 x-intercept on graph	
$b^2 - 4ac < 0$	2 complex solutions No x-intercept	

The last row of the table shows us when the parabolas never intersect the x-axis. Using the Quadratic Formula to solve the quadratic equation, the radicand is a negative. We get two complex solutions.

In the next example, the quadratic inequality solutions will result from the solution of the quadratic equation being complex.

Example:
Exercise:

Problem: Solve, writing any solution in interval notation:

ⓐ $x^2 - 3x + 4 > 0$ ⓑ $x^2 - 3x + 4 \leq 0$

Solution:

ⓐ

Write the quadratic inequality in standard form.	$-x^2 - 3x + 4 > 0$
Determine the critical points by solving the related quadratic equation.	$x^2 - 3x + 4 = 0$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c .	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{3 \pm \sqrt{-7}}{2}$
Simplify the radicand.	$x = \frac{3 \pm \sqrt{7} i}{2}$
The complex solutions tell us the	Complex solutions

parabola does not intercept the *x*-axis. Also, the parabola opens upward. This
tells us that the parabola is completely above the *x*-axis.

We are to find the solution to $x^2 - 3x + 4 > 0$. Since for all values of x the graph is above the *x*-axis, all values of *x* make the inequality true. In interval notation we write $(-\infty, \infty)$.

ⓑ

Write the quadratic inequality in standard form.	$x^2 - 3x + 4 \leq 0$
Determine the critical points by solving the related quadratic equation	$x^2 - 3x + 4 = 0$

Since the corresponding quadratic equation is the same as in part (a), the parabola will be the same. The parabola opens upward and is completely above the *x*-axis—no part of it is below the *x*-axis.

We are to find the solution to $x^2 - 3x + 4 \leq 0$. Since for all values of *x* the graph is never below the *x*-axis, no values of *x* make the inequality true. There is no solution to the inequality.

Note:
Exercise:

Problem: Solve and write any solution in interval notation: ⓐ $-x^2 + 2x - 4 \leq 0$ ⓑ $-x^2 + 2x - 4 \geq 0$

Solution:

ⓐ $(-\infty, \infty)$
ⓑ no solution

Note:
Exercise:

Solve and write any solution in interval notation:

Problem: ⓐ $x^2+3x+3<0$ ⓑ $x^2+3x+3>0$

Solution:

ⓐ no solution
ⓑ $(-\infty,\infty)$

Key Concepts

- Solve a Quadratic Inequality Graphically

  Write the quadratic inequality in standard form.
  Graph the function $f(x)=ax^2+bx+c$ using properties or transformations.
  Determine the solution from the graph.

- How to Solve a Quadratic Inequality Algebraically

  Write the quadratic inequality in standard form.
  Determine the critical points -- the solutions to the related quadratic equation.
  Use the critical points to divide the number line into intervals.
  Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  Determine the intervals where the inequality is correct. Write the solution in interval notation.

Section Exercises

Practice Makes Perfect

Solve Quadratic Inequalities Graphically

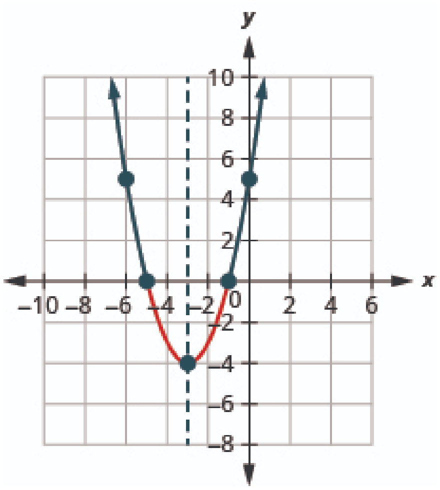
In the following exercises, ⓐ solve graphically and ⓑ write the solution in interval notation.

Exercise:

Problem: $x^2 + 6x + 5 > 0$

Solution:

ⓐ



ⓑ $(-\infty, -5) \cup (-1, \infty)$

Exercise:

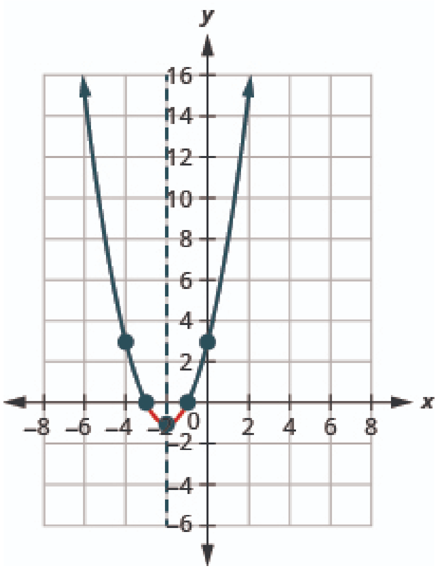
Problem: $x^2 + 4x - 12 < 0$

Exercise:

Problem: $x^2 + 4x + 3 \leq 0$

Solution:

ⓐ



ⓑ $[-3, -1]$

Exercise:

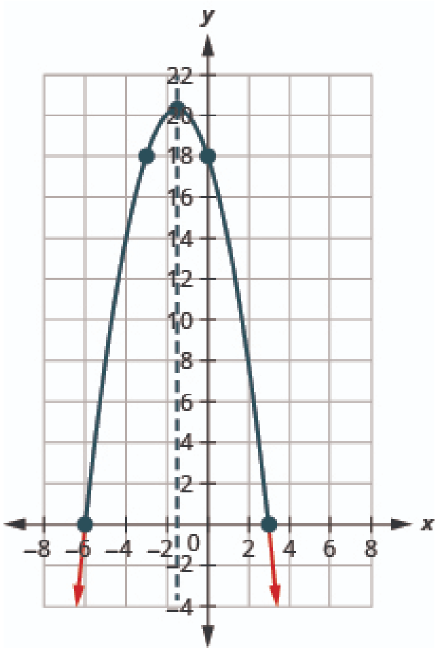
Problem: $x^2 - 6x + 8 \geq 0$

Exercise:

Problem: $-x^2 - 3x + 18 \leq 0$

Solution:

ⓐ



ⓑ $(-\infty, -6] \cup [3, \infty)$

Exercise:

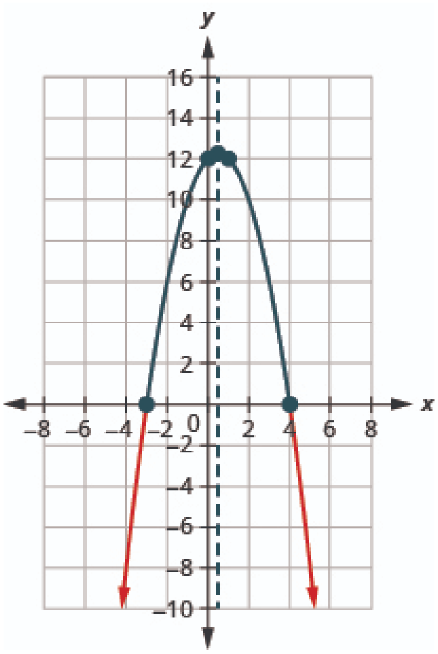
Problem: $-x^2 + 2x + 24 < 0$

Exercise:

Problem: $-x^2 + x + 12 \geq 0$

Solution:

ⓐ



ⓑ $[-3, 4]$

Exercise:

Problem: $-x^2 + 2x + 15 > 0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

Exercise:

Problem: $x^2 + 3x - 4 \geq 0$

Solution:

$(-\infty, -4] \cup [1, \infty)$

Exercise:

Problem: $x^2 + x - 6 \leq 0$

Exercise:

Problem: $x^2 - 7x + 10 < 0$

Solution:

$(2, 5)$

Exercise:

Problem: $x^2 - 4x + 3 > 0$

Exercise:

Problem: $x^2 + 8x > -15$

Solution:

$(-\infty, -5) \cup (-3, \infty)$

Exercise:

Problem: $x^2 + 8x < -12$

Exercise:

Problem: $x^2 - 4x + 2 \leq 0$

Solution:

$\left[2 - \sqrt{2}, 2 + \sqrt{2}\right]$

Exercise:

Problem: $-x^2 + 8x - 11 < 0$

Exercise:

Problem: $x^2 - 10x > -19$

Solution:

$\left(-\infty, 5 - \sqrt{6}\right) \cup \left(5 + \sqrt{6}, \infty\right)$

Exercise:

Problem: $x^2 + 6x < -3$

Exercise:

Problem: $-6x^2 + 19x - 10 \geq 0$

Solution:

$\left(-\infty, -\frac{5}{2}\right] \cup \left[-\frac{2}{3}, \infty\right)$

Exercise:

Problem: $-3x^2 - 4x + 4 \leq 0$

Exercise:

Problem: $-2x^2 + 7x + 4 \geq 0$

Solution:

$\left(-\infty, -\frac{1}{2}\right] \cup [4, \infty)$

Exercise:

Problem: $2x^2 + 5x - 12 > 0$

Exercise:

Problem: $x^2 + 3x + 5 > 0$

Solution:

$(-\infty, \infty).$

Exercise:

Problem: $x^2 - 3x + 6 \leq 0$

Exercise:

Problem: $-x^2 + x - 7 > 0$

Solution:

no solution

Exercise:

Problem: $-x^2 - 4x - 5 < 0$

Exercise:

Problem: $-2x^2 + 8x - 10 < 0$

Solution:

$(-\infty, \infty)$.

Exercise:

Problem: $-x^2 + 2x - 7 \geq 0$

Writing Exercises

Exercise:

Problem:

Explain critical points and how they are used to solve quadratic inequalities algebraically.

Solution:

Answers will vary.

Exercise:

Problem:

Solve $x^2 + 2x \geq 8$ both graphically and algebraically. Which method do you prefer, and why?

Exercise:

Problem:

Describe the steps needed to solve a quadratic inequality graphically.

Solution:

Answers will vary.

Exercise:

Problem:

Describe the steps needed to solve a quadratic inequality algebraically.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic inequalities graphically.			
solve quadratic inequalities algebraically.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $y^2 = 144$

Solution:

$y = \pm 12$

Exercise:

Problem: $n^2 - 80 = 0$

Exercise:

Problem: $4a^2 = 100$

Solution:

$a = \pm 5$

Exercise:

Problem: $2b^2 = 72$

Exercise:

Problem: $r^2 + 32 = 0$

Solution:

$r = \pm 4\sqrt{2}i$

Exercise:

Problem: $t^2 + 18 = 0$

Exercise:

Problem: $\frac{2}{3}w^2 - 20 = 30$

Solution:

$w = \pm 5\sqrt{3}$

Exercise:

Problem: $11.\ 5c^2 + 3 = 19$

Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $(p-5)^2 + 3 = 19$

Solution:

$p = -1, 9$

Exercise:

Problem: $(u+1)^2 = 45$

Exercise:

Problem: $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$

Solution:

$x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$

Exercise:

Problem: $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

Exercise:

Problem: $(n-4)^2-50=150$

Solution:

$n=4\pm 10\sqrt{2}$

Exercise:

Problem: $(4c-1)^2=-18$

Exercise:

Problem: $n^2+10n+25=12$

Solution:

$n=-5\pm 2\sqrt{3}$

Exercise:

Problem: $64a^2+48a+9=81$

Solve Quadratic Equations by Completing the Square

Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

Exercise:

Problem: x^2+22x

Solution:

$(x+11)^2$

Exercise:

Problem: $m^2 - 8m$

Exercise:

Problem: $a^2 - 3a$

Solution:

$\left(a - \frac{3}{2}\right)^2$

Exercise:

Problem: $b^2 + 13b$

In the following exercises, solve by completing the square.

Exercise:

Problem: $d^2 + 14d = -13$

Solution:

$d = -13, -1$

Exercise:

Problem: $y^2 - 6y = 36$

Exercise:

Problem: $m^2 + 6m = -109$

Solution:

$m = -3 \pm 10i$

Exercise:

Problem: $t^2 - 12t = -40$

Exercise:

Problem: $v^2 - 14v = -31$

Solution:

$v = 7 \pm 3\sqrt{2}$

Exercise:

Problem: $w^2 - 20w = 100$

Exercise:

Problem: $m^2 + 10m - 4 = -13$

Solution:

$m = -9, -1$

Exercise:

Problem: $n^2 - 6n + 11 = 34$

Exercise:

Problem: $a^2 = 3a + 8$

Solution:

$a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$

Exercise:

Problem: $b^2 = 11b - 5$

Exercise:

Problem: $(u+8)(u+4)=14$

Solution:

$u=-6\pm 2\sqrt{2}$

Exercise:

Problem: $(z-10)(z+2)=28$

Solve Quadratic Equations of the form $ax^2+bx+c=0$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $3p^2-18p+15=15$

Solution:

$p=0,6$

Exercise:

Problem: $5q^2+70q+20=0$

Exercise:

Problem: $4y^2-6y=4$

Solution:

$y=-\frac{1}{2},2$

Exercise:

Problem: $2x^2+2x=4$

Exercise:

Problem: $3c^2 + 2c = 9$

Solution:

$c = -\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$

Exercise:

Problem: $4d^2 - 2d = 8$

Exercise:

Problem: $2x^2 + 6x = -5$

Solution:

$x = \frac{3}{2} \pm \frac{1}{2}i$

Exercise:

Problem: $2x^2 + 4x = -5$

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

Exercise:

Problem: $4x^2 - 5x + 1 = 0$

Solution:

$x = \frac{1}{4}, 1$

Exercise:

Problem: $7y^2 + 4y - 3 = 0$

Exercise:

Problem: $r^2 - r - 42 = 0$

Solution:

$r = -6, 7$

Exercise:

Problem: $t^2 + 13t + 22 = 0$

Exercise:

Problem: $4v^2 + v - 5 = 0$

Solution:

$v = \frac{-1 \pm \sqrt{21}}{8}$

Exercise:

Problem: $2w^2 + 9w + 2 = 0$

Exercise:

Problem: $3m^2 + 8m + 2 = 0$

Solution:

$m = \frac{-4 \pm \sqrt{10}}{3}$

Exercise:

Problem: $5n^2 + 2n - 1 = 0$

Exercise:

Problem: $6a^2 - 5a + 2 = 0$

Solution:

$a = \frac{5}{12} \pm \frac{\sqrt{23}}{12} i$

Exercise:

Problem: $4b^2 - b + 8 = 0$

Exercise:

Problem: $u\left(u - 10\right) + 3 = 0$

Solution:

$u = 5 \pm \sqrt{21}$

Exercise:

Problem: $5z(z - 2) = 3$

Exercise:

Problem: $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

Solution:

$p = \frac{4 \pm \sqrt{5}}{5}$

Exercise:

Problem: $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

Exercise:

Problem: $4c^2 + 4c + 1 = 0$

Solution:

$c = -\frac{1}{2}$

Exercise:

Problem: $9d^2 - 12d = -4$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

Exercise:

Problem:
ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

Solution:

ⓐ 1 ⓑ 2 ⓒ 2 ⓓ 2

Exercise:

Problem:
ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

Exercise:

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
Problem: ⓒ $3(c+2)^2 = 15$

Solution:

ⓐ factor ⓑ Quadratic Formula ⓒ square root

Exercise:

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
Problem: ⓒ $6(5v-7)^2 = 150$

Solve Equations in Quadratic Form

Solve Equations in Quadratic Form

In the following exercises, solve.

Exercise:

Problem: $x^4 - 14x^2 + 24 = 0$

Solution:

$x = \pm\sqrt{2},\ \ x = \pm 2\sqrt{3}$

Exercise:

Problem: $x^4 + 4x^2 - 32 = 0$

Exercise:

Problem: $4x^4 - 5x^2 + 1 = 0$

Solution:

$x = \pm 1,\ \ x = \pm \frac{1}{2}$

Exercise:

Problem: $\left(2y+3\right)^2 + 3\left(2y+3\right) - 28 = 0$

Exercise:

Problem: $x + 3\sqrt{x} - 28 = 0$

Solution:

$x = 16$

Exercise:

Problem: $6x + 5\sqrt{x} - 6 = 0$

Exercise:

Problem: $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 24 = 0$

Solution:

$x = 64, x = 216$

Exercise:

Problem: $x + 7x^{\frac{1}{2}} + 6 = 0$

Exercise:

Problem: $8x^{-2} - 2x^{-1} - 3 = 0$

Solution:

$x = -2, x = \frac{4}{3}$

Solve Applications of Quadratic Equations

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

Exercise:

Problem: Find two consecutive odd numbers whose product is 323.

Exercise:

Problem: Find two consecutive even numbers whose product is 624.

Solution:

Two consecutive even numbers whose product is 624 are 24 and 26, and −24 and −26.

Exercise:

Problem:

A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

Exercise:

Problem:

Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

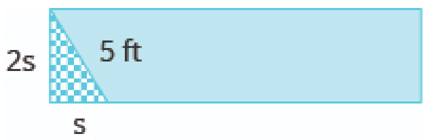
Solution:

The height is 14 inches and the width is 10 inches.

Exercise:

Problem:

A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



Exercise:

Problem:

A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

Solution:

The length of the diagonal is 3.6 feet.

Exercise:

Problem:

The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

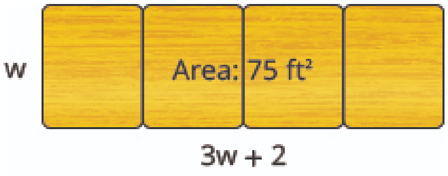
Exercise:

Problem:

For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

Solution:

The width of the serving table is 4.7 feet and the length is 16.1 feet.



Exercise:

Problem:

A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

Exercise:

Problem:

The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

Solution:

The speed of the wind was 30 mph.

Exercise:

Problem:

Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

Exercise:

Problem:

Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Solution:

One man takes 3 hours and the other man 6 hours to finish the repair alone.

Graph Quadratic Functions Using Properties

Recognize the Graph of a Quadratic Function

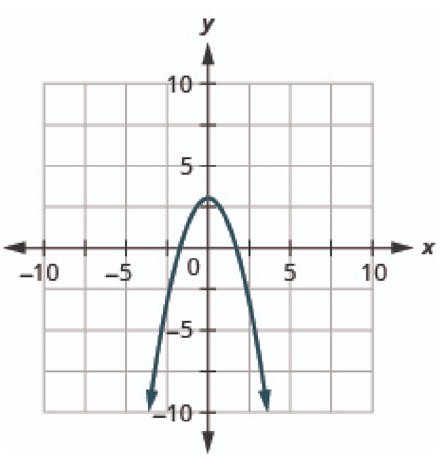
In the following exercises, graph by plotting point.
Exercise:

Problem: Graph $y = x^2 - 2$

Exercise:

Problem: Graph $y = -x^2 + 3$

Solution:



In the following exercises, determine if the following parabolas open up or down.
Exercise:

Problem: ⓐ $y = -3x^2 + 3x - 1$
ⓑ $y = 5x^2 + 6x + 3$

Exercise:

Problem: ⓐ $y = x^2 + 8x - 1$
ⓑ $y = -4x^2 - 7x + 1$

Solution:

ⓐ up ⓑ down

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the equation of the axis of symmetry and ⓑ the vertex.

Exercise:

Problem: $y = -x^2 + 6x + 8$

Exercise:

Problem: $y = 2x^2 - 8x + 1$

Solution:

$x = 2; (2, -7)$

Find the Intercepts of a Parabola

In the following exercises, find the *x*- and *y*-intercepts.

Exercise:

Problem: $y = x^2 - 4x + 5$

Exercise:

Problem: $y = x^2 - 8x + 15$

Solution:

$y : (0, 15)$
$x : (3, 0), (5, 0)$

Exercise:

Problem: $y = x^2 - 4x + 10$

Exercise:

Problem: $y = -5x^2 - 30x - 46$

Solution:

$y : (0, -46)$
$x : \text{none}$

Exercise:

Problem: $y = 16x^2 - 8x + 1$

Exercise:

Problem: $y = x^2 + 16x + 64$

Solution:

$y : (0, -64)$
$x : (-8, 0)$

Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

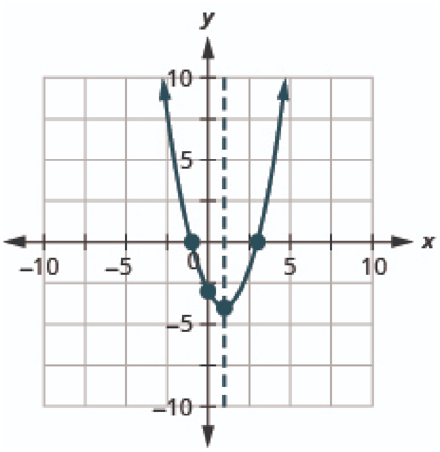
Exercise:

Problem: $y = x^2 + 8x + 15$

Exercise:

Problem: $y = x^2 - 2x - 3$

Solution:



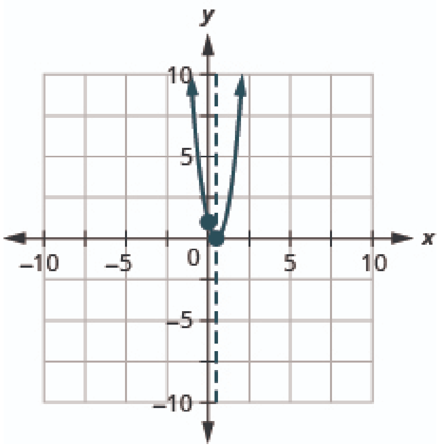
Exercise:

Problem: $y = -x^2 + 8x - 16$

Exercise:

Problem: $y = 4x^2 - 4x + 1$

Solution:



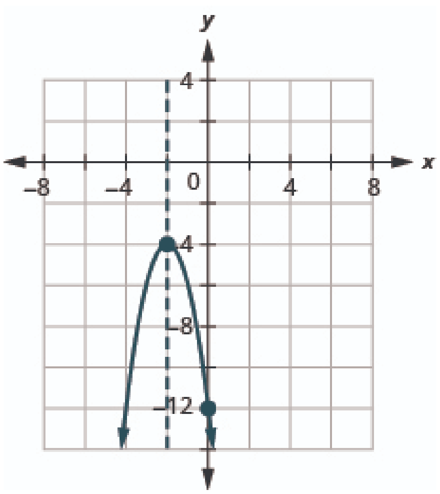
Exercise:

Problem: $y=x^2+6x+13$

Exercise:

Problem: $y=-2x^2-8x-12$

Solution:



Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

Exercise:

Problem: $y = 7x^2 + 14x + 6$

Exercise:

Problem: $y = -3x^2 + 12x - 10$

Solution:

The maximum value is 2 when $x = 2$.

In the following exercises, solve. Rounding answers to the nearest tenth.

Exercise:

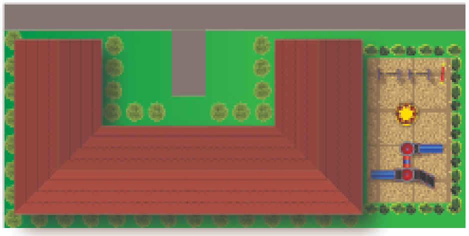
Problem:

A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

Exercise:

Problem:

A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



Solution:

The length adjacent to the building is 90 feet giving a maximum area of 4,050 square feet.

Graph Quadratic Functions Using Transformations

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

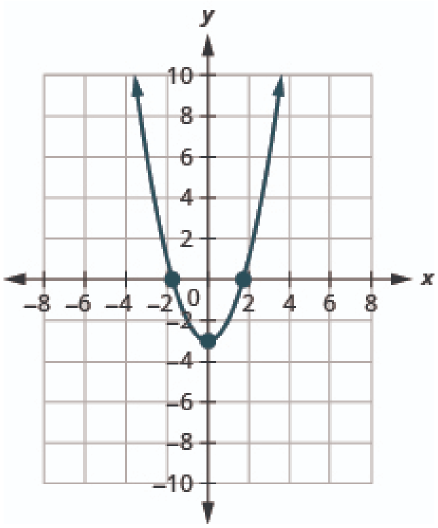
Exercise:

Problem: $g\left(x\right) = x^2 + 4$

Exercise:

Problem: $h\left(x\right) = x^2 - 3$

Solution:



In the following exercises, graph each function using a horizontal shift.

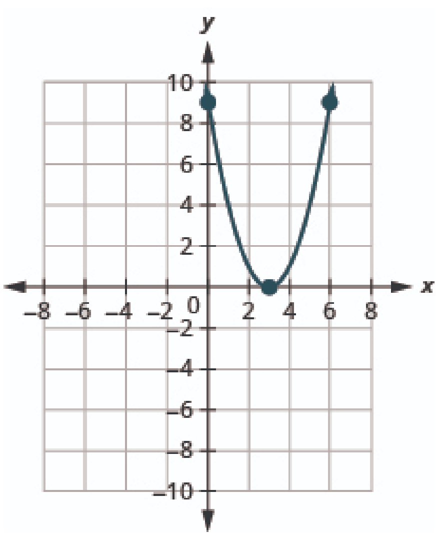
Exercise:

Problem: $f(x) = (x+1)^2$

Exercise:

Problem: $g(x) = (x-3)^2$

Solution:



In the following exercises, graph each function using transformations.

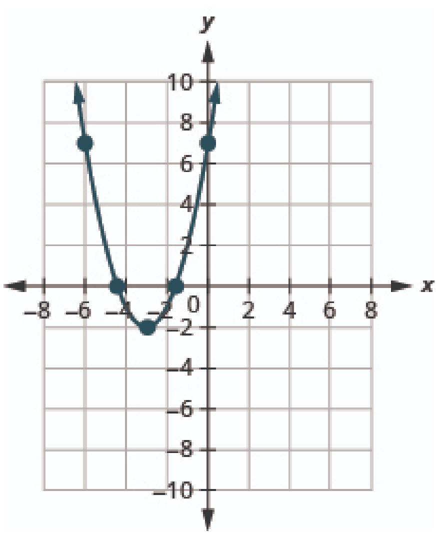
Exercise:

Problem: $f(x) = (x + 2)^2 + 3$

Exercise:

Problem: $f(x) = (x + 3)^2 - 2$

Solution:



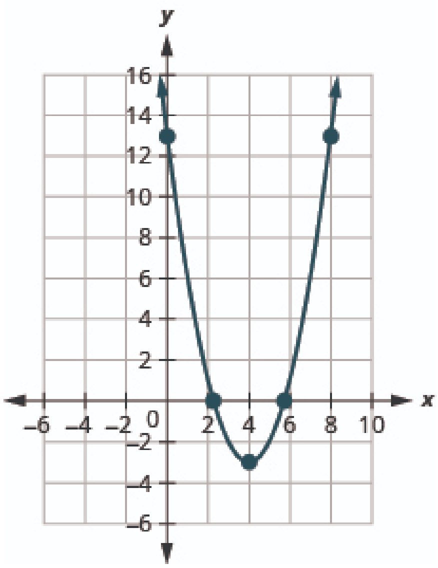
Exercise:

Problem: $f(x) = (x - 1)^2 + 4$

Exercise:

Problem: $f(x) = (x - 4)^2 - 3$

Solution:



Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

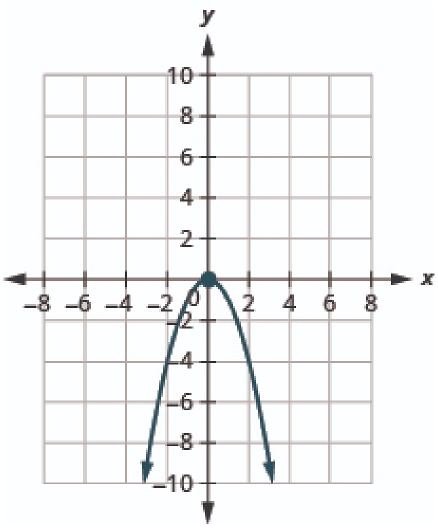
Exercise:

Problem: $f(x) = 2x^2$

Exercise:

Problem: $f(x) = -x^2$

Solution:



Exercise:

Problem: $f(x) = \frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f\left(x\right)=a{\left(x-h\right)}^{2}+k$ form by completing the square.

Exercise:

Problem: $f\left(x\right)=2x^2-4x-4$

Solution:

$f\left(x\right)=2{\left(x-1\right)}^{2}-6$

Exercise:

Problem: $f\left(x\right)=3x^2+12x+8$

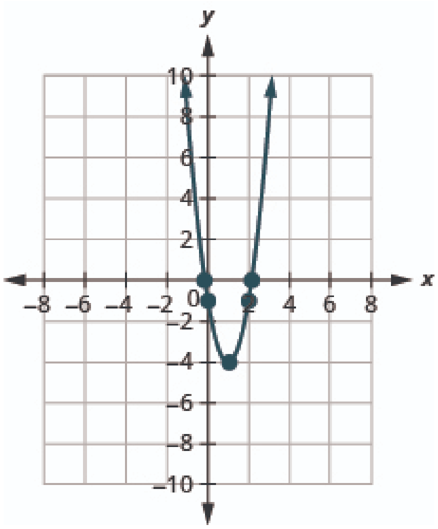
In the following exercises, ⓐ rewrite each function in $f\left(x\right)=a{\left(x-h\right)}^{2}+k$ form and ⓑ graph it by using transformations.

Exercise:

Problem: $f\left(x\right)=3x^2-6x-1$

Solution:

ⓐ $f\left(x\right)=3(x-1)^2-4$
ⓑ



Exercise:

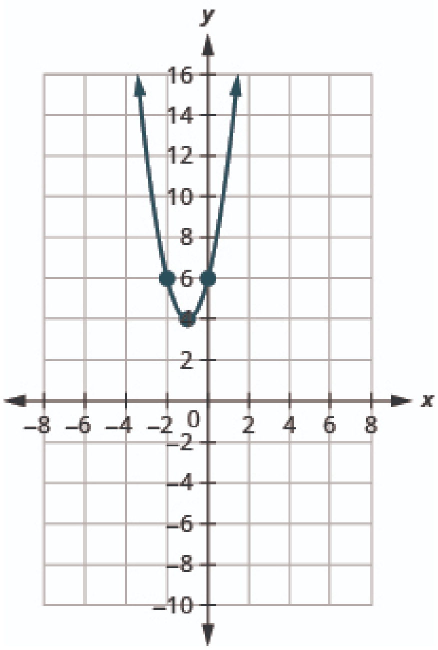
Problem: $f\left(x\right)=-2x^2-12x-5$

Exercise:

Problem: $f\left(x\right)=2x^2+4x+6$

Solution:

ⓐ $f\left(x\right)=2(x+1)^2+4$
ⓑ



Exercise:

Problem: $f(x) = 3x^2 - 12x + 7$

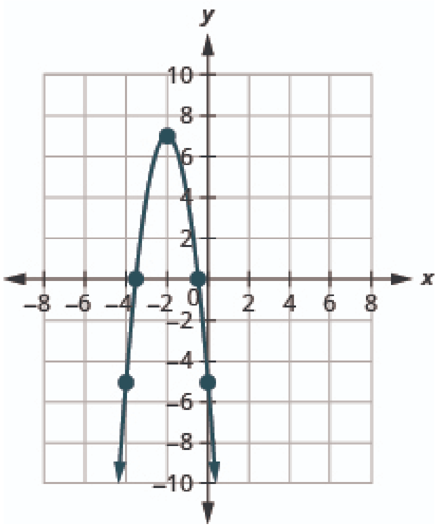
In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)^2 + k$ form and ⓑ graph it using properties.

Exercise:

Problem: $f(x) = -3x^2 - 12x - 5$

Solution:

ⓐ $f(x) = -3(x+2)^2 + 7$
ⓑ



Exercise:

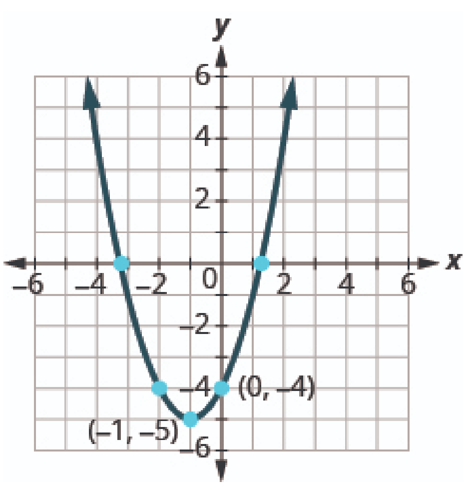
Problem: $f\left(x\right)=2{x}^{2}-12x+7$

Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f\left(x\right)=a{\left(x-h\right)}^{2}+k$ form.

Exercise:

Problem:

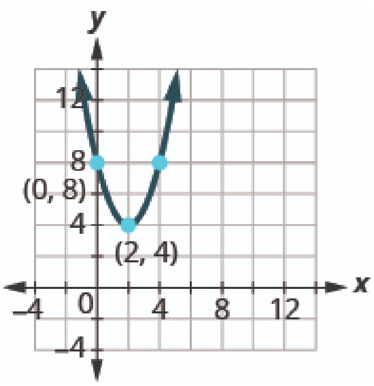


Solution:

$f(x) = (x+1)^2 - 5$

Exercise:

Problem:



Solve Quadratic Inequalities

Solve Quadratic Inequalities Graphically

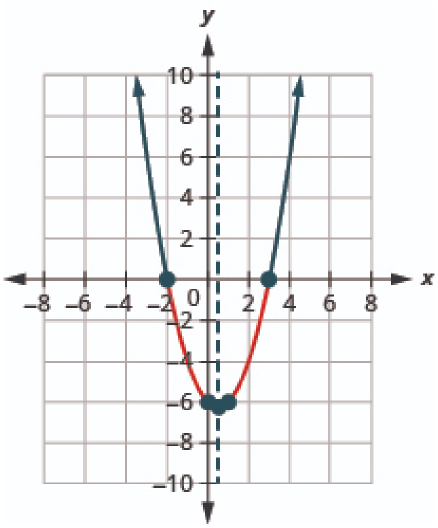
In the following exercises, solve graphically and write the solution in interval notation.

Exercise:

Problem: $x^2 - x - 6 > 0$

Solution:

ⓐ



ⓑ $(-\infty, -2) \cup (3, \infty)$

Exercise:

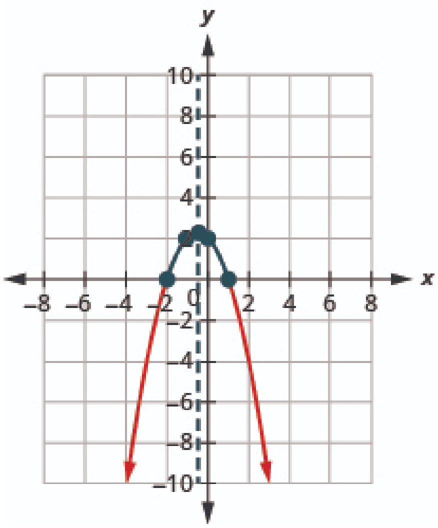
Problem: $x^2 + 4x + 3 \leq 0$

Exercise:

Problem: $-x^2 - x + 2 \geq 0$

Solution:

ⓐ



$[-2,1]$

Exercise:

Problem: $-x^2+2x+3<0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

Exercise:

Problem: $x^2-6x+8<0$

Solution:

$(2,4)$

Exercise:

Problem: $x^2+x>12$

Exercise:

Problem: $x^2-6x+4\leq 0$

Solution:

$\left[3-\sqrt{5}, 3+\sqrt{5}\right]$

Exercise:

Problem: $2x^2+7x-4>0$

Exercise:

Problem: $-x^2+x-6>0$

Solution:

no solution

Exercise:

Problem: $x^2-2x+4\geq 0$

Practice Test

Exercise:

Problem:

Use the Square Root Property to solve the quadratic equation $3(w+5)^2=27$.

Solution:

$w=-2, w=-8$

Exercise:

Problem:

Use Completing the Square to solve the quadratic equation $a^2-8a+7=23$.

Exercise:

Problem:

Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Solution:

$m = 1, m = \frac{3}{2}$

Solve the following quadratic equations. Use any method.

Exercise:

Problem: $2x\left(3x - 2\right) - 1 = 0$

Exercise:

Problem: $\frac{9}{4}y^2 - 3y + 1 = 0$

Solution:

$y = \frac{2}{3}$

Use the discriminant to determine the number and type of solutions of each quadratic equation.

Exercise:

Problem: $6p^2 - 13p + 7 = 0$

Exercise:

Problem: $3q^2 - 10q + 12 = 0$

Solution:

2 complex

Solve each equation.

Exercise:

Problem: $4x^4 - 17x^2 + 4 = 0$

Exercise:

Problem: $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

Solution:

$y = 1, y = -27$

For each parabola, find ⓐ which direction it opens, ⓑ the equation of the axis of symmetry, ⓒ the vertex, ⓓ the *x*- and *y*-intercepts, and e) the maximum or minimum value.

Exercise:

Problem: $y = 3x^2 + 6x + 8$

Exercise:

Problem: $y = -x^2 - 8x + 16$

Solution:

ⓐ down ⓑ $x = -4$
ⓒ $(-4, 0)$ ⓓ y: $(0, 16)$; x: $(-4, 0)$
ⓔ minimum value of -4 when $x = 0$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

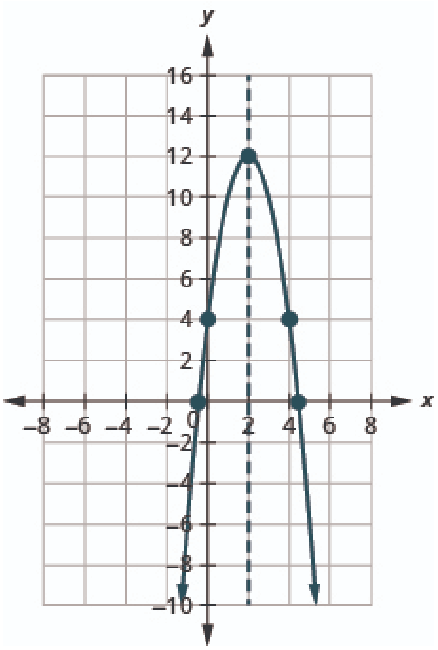
Exercise:

Problem: $f(x) = x^2 + 6x + 9$

Exercise:

Problem: $f(x) = -2x^2 + 8x + 4$

Solution:



In the following exercises, graph each function using transformations.

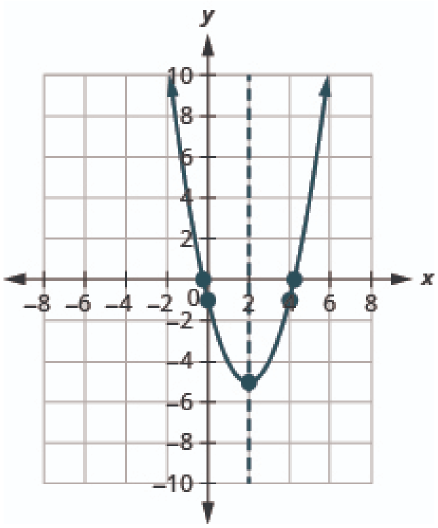
Exercise:

Problem: $f(x) = (x+3)^2 + 2$

Exercise:

Problem: $f\left(x\right) = x^2 - 4x - 1$

Solution:



$f\left(x\right)=2(x-1)^{2}-6$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

Exercise:

Problem: $x^{2}-6x-8\le 0$

Exercise:

Problem: $2x^{2}+x-10>0$

Solution:

$\left(-\infty,-\frac{5}{2}\right)\cup\left(2,\infty\right)$

Model the situation with a quadratic equation and solve by any method.

Exercise:

Problem: Find two consecutive even numbers whose product is 360.

Exercise:

Problem:

The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Solution:

The diagonal is 3.8 units long.

Exercise:

Problem:

A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

Glossary

quadratic inequality
: A quadratic inequality is an inequality that contains a quadratic expression.

Introduction
class="introduction"

Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

Finding Composite and Inverse Functions
By the end of this section, you will be able to:

- Find and evaluate composite functions
- Determine whether a function is one-to-one
- Find the inverse of a function

Note:
Before you get started, take this readiness quiz.

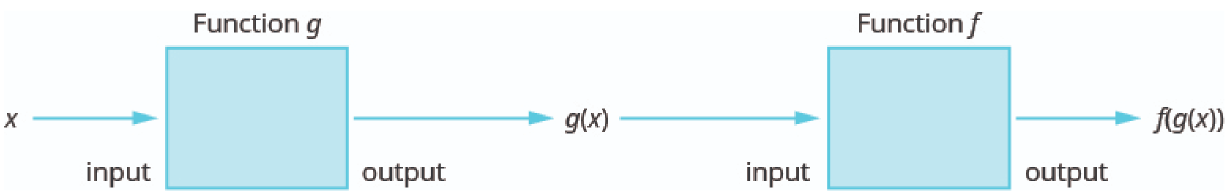
1. If $f\left(x\right)=2x-3$ and $g\left(x\right)={x}^{2}+2x-3$, find $f\left(4\right)$.
   If you missed this problem, review [link].
2. Solve for x, $3x+2y=12$.
   If you missed this problem, review [link].
3. Simplify: $5\frac{\left(x+4\right)}{5}-4$.
   If you missed this problem, review [link].

In this chapter, we will introduce two new types of functions, exponential functions and logarithmic functions. These functions are used extensively in business and the sciences as we will see.

Find and Evaluate Composite Functions

Before we introduce the functions, we need to look at another operation on functions called composition. In composition, the output of one function is the input of a second function. For functions f and g, the composition is written $f\circ g$ and is defined by $\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$.

We read $f\left(g\left(x\right)\right)$ as "f of g of x."



To do a composition, the output of the first function, $g\left(x\right)$, becomes the input of the second function, *f*, and so we must be sure that it is part of the domain of *f*.

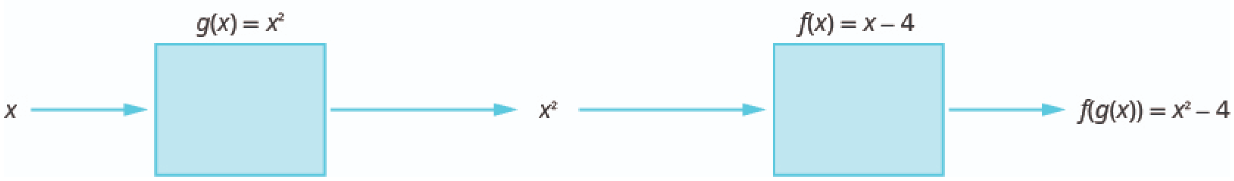
Note:
Composition of Functions
The composition of functions *f* and *g* is written $f\cdot g$ and is defined by

Equation:

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of x.

We have actually used composition without using the notation many times before. When we graphed quadratic functions using translations, we were composing functions. For example, if we first graphed $g(x) = x^2$ as a parabola and then shifted it down vertically four units, we were using the composition defined by $(f \circ g)(x) = f(g(x))$ where $f(x) = x - 4$.



The next example will demonstrate that $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \cdot g)(x)$ usually result in different outputs.

Example:
Exercise:

Problem:

For functions $f(x) = 4x - 5$ and $g(x) = 2x + 3$, find: ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution:

ⓐ

Use the definition of $(f \circ g)(x)$.	$(f \circ g)(x) = f(g(x))$
Substitute $2x + 3$ for $g(x)$.	$(f \circ g)(x) = f(2x + 3)$

Find $f(2x+3)$ where $f(x) = 4x - 5$.	$(f \circ g)(x) = 4(2x+3) - 5$
Distribute.	$(f \circ g)(x) = 8x + 12 - 5$
Simplify.	$(f \circ g)(x) = 8x + 7$

ⓑ

Use the definition of $(f \circ g)(x)$.	$(g \circ f)(x) = g(f(x))$
Substitute $4x - 5$ for $f(x)$.	$(g \circ f)(x) = g(4x - 5)$
Find $g(4x - 5)$ where $g(x) = 2x + 3$.	$(g \circ f)(x) = 2(4x - 5) + 3$
Distribute.	$(g \circ f)(x) = 8x - 10 + 3$
Simplify.	$(g \circ f)(x) = 8x - 7$

Notice the difference in the result in part ⓐ and part ⓑ.

ⓒ Notice that $(f \cdot g)(x)$ is different than $(f \circ g)(x)$. In part ⓐ we did the composition of the functions. Now in part ⓒ we are not composing them, we are multiplying them.

Use the definition of $(f \cdot g)(x)$.	$(f \cdot g)(x) = f(x) \cdot g(x)$
Substitute $f(x) = 4x - 5$ and $g(x) = 2x + 3$.	$(f \cdot g)(x) = (4x - 5) \cdot (2x + 3)$
Multiply.	$(f \cdot g)(x) = 8x^2 + 2x - 15$

Note:
Exercise:

Problem:

For functions $f\left(x\right)=3x-2$ and $g\left(x\right)=5x+1$, find ⓐ $\left(f\circ g\right)\left(x\right)$ ⓑ $\left(g\circ f\right)\left(x\right)$ ⓒ $\left(f\cdot g\right)\left(x\right)$.

Solution:

ⓐ $15x+1$ ⓑ $15x-9$
ⓒ $15x^2-7x-2$

Note:
Exercise:

Problem:

For functions $f\left(x\right)=4x-3$, and $g\left(x\right)=6x-5$, find ⓐ $\left(f\circ g\right)\left(x\right)$, ⓑ $\left(g\circ f\right)\left(x\right)$, and ⓒ $\left(f\cdot g\right)\left(x\right)$.

Solution:

ⓐ $24x-23$ ⓑ $24x-23$
ⓒ $24x^2-38x+15$

In the next example we will evaluate a composition for a specific value.

Example:
Exercise:

Problem:

For functions $f\left(x\right)=x^2-4$, and $g\left(x\right)=3x+2$, find: ⓐ $\left(f\circ g\right)\left(-3\right)$, ⓑ $(g\circ f)(-1)$, and ⓒ $(f\circ f)(2)$.

Solution:

ⓐ

Use the definition of $(f \circ g)(-3)$.	$(f \circ g)(-3) = f(g(-3))$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(f \circ g)(-3) = f(3 \cdot (-3) + 2)$
Simplify.	$(f \circ g)(-3) = f(-7)$
Find $f(-7)$ where $f(x) = x^2 - 4$.	$(f \circ g)(-3) = (-7)^2 - 4$
Simplify.	$(f \circ g)(-3) = 45$

ⓑ

Use the definition of $(g \circ f)(-1)$.	$(g \circ f)(-1) = g(f(-1))$
Find $f(-1)$ where $f(x) = x^2 - 4$.	$(g \circ f)(-1) = g((-1)^2 - 4)$
Simplify.	$(g \circ f)(-1) = g(-3)$
Find $g(-3)$ where $g(x) = 3x + 2$.	$(g \circ f)(-1) = 3(-3) + 2$
Simplify.	$(g \circ f)(-1) = -7$

ⓒ

Use the definition of $(f \circ f)(2)$.	$(f \circ f)(2) = f(f(2))$
Find $f(2)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = f(2^2 - 4)$
Simplify.	$(f \circ f)(2) = f(0)$
Find $f(0)$ where $f(x) = x^2 - 4$.	$(f \circ f)(2) = 0^2 - 4$
Simplify.	$(f \circ f)(2) = -4$

Note:
Exercise:

Problem:

For functions $f(x) = x^2 - 9$, and $g(x) = 2x + 5$, find ⓐ $(f \circ g)(-2)$, ⓑ $(g \circ f)(-3)$, and ⓒ $(f \circ f)(4)$.

Solution:

ⓐ –8 ⓑ 5 ⓒ 40

Note:
Exercise:

Problem:

For functions $f(x) = x^2 + 1$, and $g(x) = 3x - 5$, find ⓐ $(f \circ g)(-1)$, ⓑ $(g \circ f)(2)$, and ⓒ $(f \circ f)(-1)$.

Solution:

ⓐ 65 ⓑ 10 ⓒ 5

Determine Whether a Function is One-to-One

When we first introduced functions, we said a function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

We used the birthday example to help us understand the definition. Every person has a birthday, but no one has two birthdays and it is okay for two people to share a birthday. Since each person has exactly one birthday, that relation is a function.

A function is **one-to-one** if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Our example of the birthday relation is not a one-to-one function. Two people can share the same birthday. The range value August 2 is the birthday of Liz and June, and so one range value has two domain values. Therefore, the function is not one-to-one.

Note:
One-to-One Function

A function is **one-to-one** if each value in the range corresponds to one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value. There are no repeated *y*-values.

Example:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and, if so, if the function is one-to-one.

ⓐ $\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$ and ⓑ $\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$.

Solution:

ⓐ

$$\{(-3,27),(-2,8),(-1,1),(0,0),(1,1),(2,8),(3,27)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

But each *y*-value is not paired with only one *x*-value, $(-3,27)$ and $(3,27)$, for example. So this function is not one-to-one.

ⓑ

$$\{(0,0),(1,1),(4,2),(9,3),(16,4)\}$$

Each *x*-value is matched with only one *y*-value. So this relation is a function.

Since each *y*-value is paired with only one *x*-value, this function is one-to-one.

Note:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(-3,-6),(-2,-4),(-1,-2),(0,0),(1,2),(2,4),(3,6)\}$
ⓑ $\{(-4,8),(-2,4),(-1,2),(0,0),(1,2),(2,4),(4,8)\}$

Solution:

ⓐ One-to-one function
ⓑ Function; not one-to-one

Note:
Exercise:

Problem:

For each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

ⓐ $\{(27, -3), (8, -2), (1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$
ⓑ $\{(7, -3), (-5, -4), (8, 0), (0, 0), (-6, 4), (-2, 2), (-1, 3)\}$

Solution:

ⓐ Not a function
ⓑ Function; not one-to-one

To help us determine whether a relation is a function, we use the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. Also, if any vertical line intersects the graph in more than one point, the graph does not represent a function.

The vertical line is representing an *x*-value and we check that it intersects the graph in only one *y*-value. Then it is a function.

To check if a function is one-to-one, we use a similar process. We use a horizontal line and check that each horizontal line intersects the graph in only one point. The horizontal line is representing a *y*-value and we check that it intersects the graph in only one *x*-value. If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function. This is the **horizontal line test**.

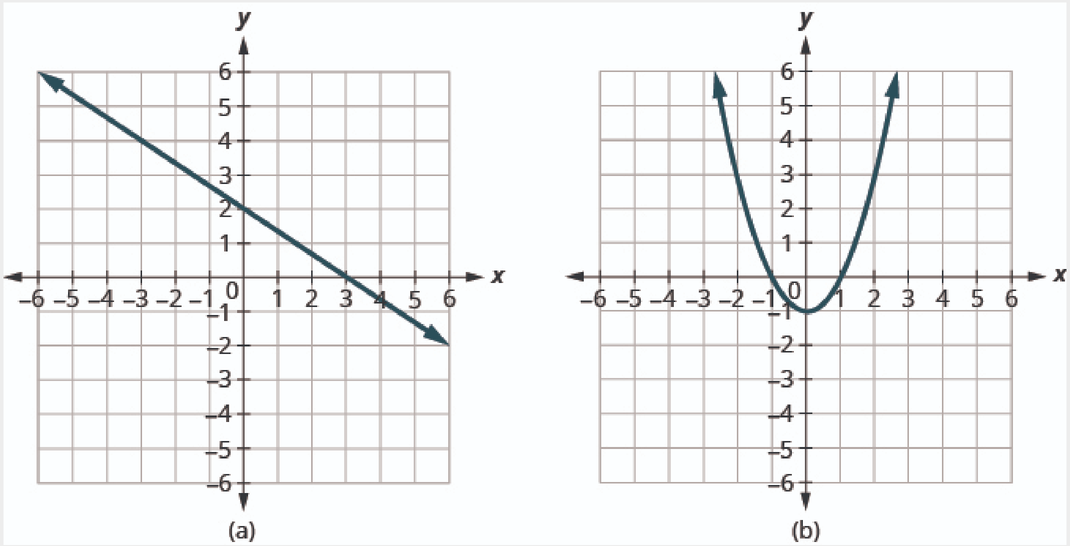
Note:
Horizontal Line Test
If every horizontal line intersects the graph of a function in at most one point, it is a one-to-one function.

We can test whether a graph of a relation is a function by using the vertical line test. We can then tell if the function is one-to-one by applying the horizontal line test.

Example:
Exercise:

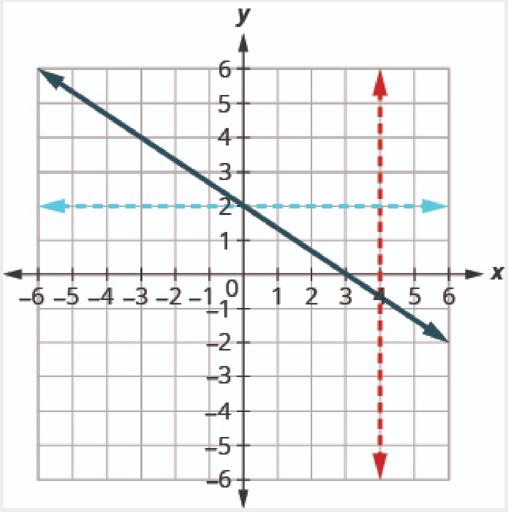
Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



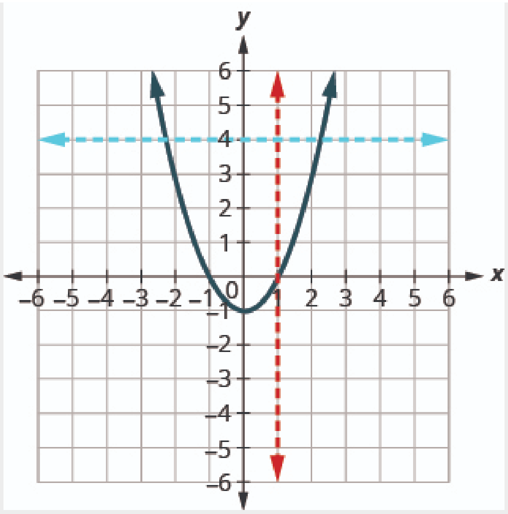
Solution:

ⓐ



Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

ⓑ

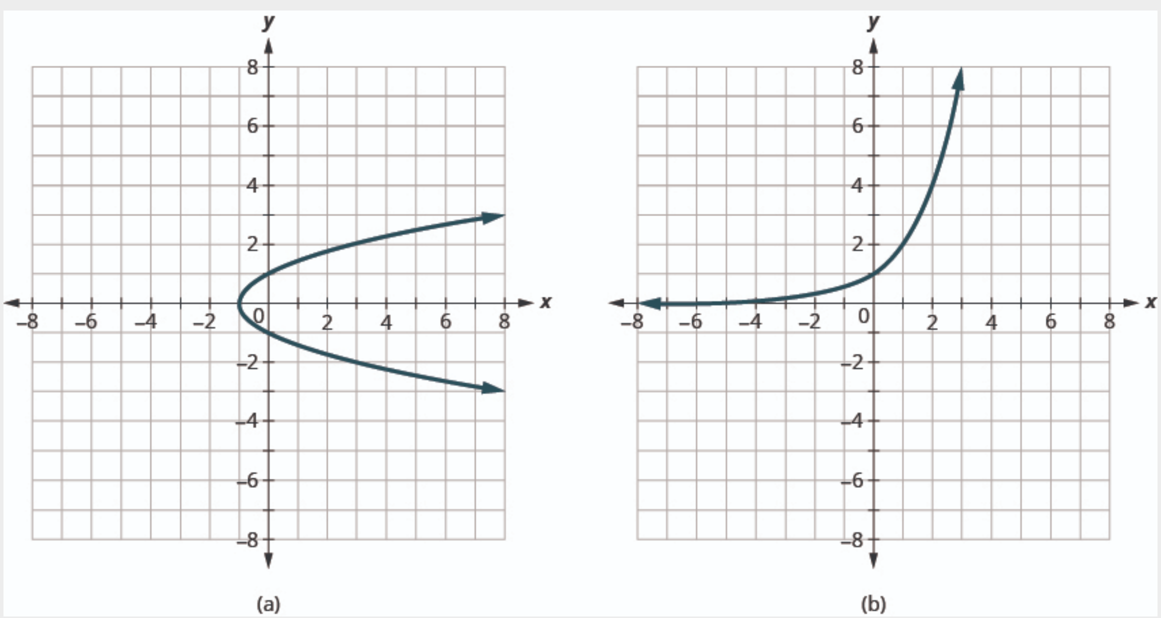


Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. The horizontal line shown on the graph intersects it in two points. This graph does not represent a one-to-one function.

Note:
Exercise:

Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.



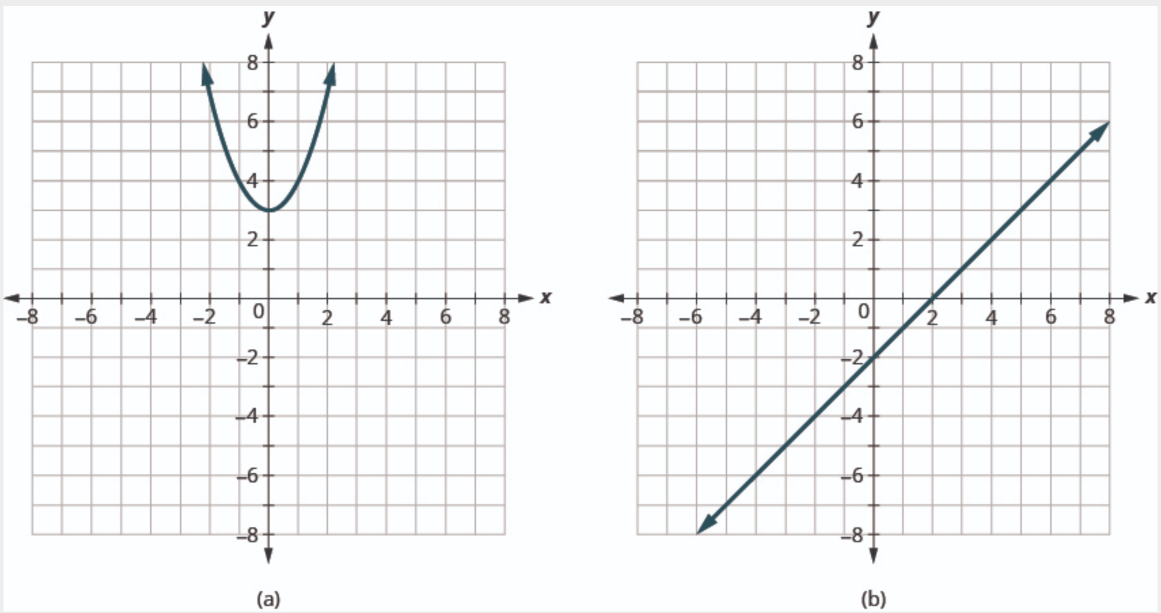
Solution:

ⓐ Not a function ⓑ One-to-one function

Note:
Exercise:

Problem:

Determine ⓐ whether each graph is the graph of a function and, if so, ⓑ whether it is one-to-one.

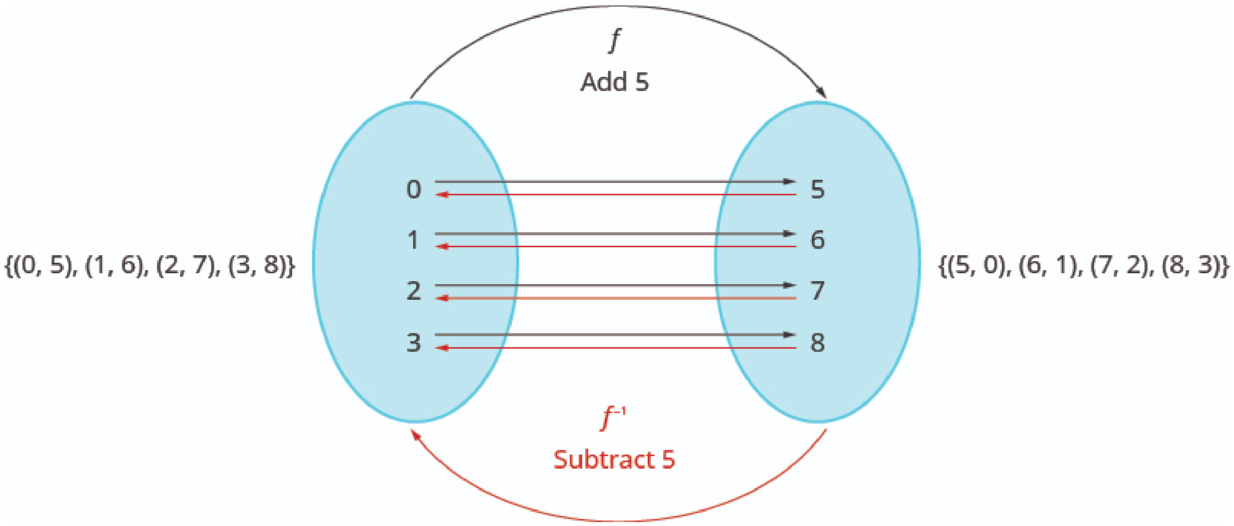


Solution:

ⓐ Function; not one-to-one ⓑ One-to-one function

Find the Inverse of a Function

Let’s look at a one-to one function, f, represented by the ordered pairs $\{(0,5),(1,6),(2,7),(3,8)\}$. For each x-value, f adds 5 to get the y-value. To ‘undo’ the addition of 5, we subtract 5 from each y-value and get back to the original x-value. We can call this “taking the inverse of f” and name the function f^{-1}.



Notice that that the ordered pairs of f and f^{-1} have their x-values and y-values reversed. The domain of f is the range of f^{-1} and the domain of f^{-1} is the range of f.

Note:
Inverse of a Function Defined by Ordered Pairs
If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).

In the next example we will find the inverse of a function defined by ordered pairs.

Example:
Exercise:

Problem:

Find the inverse of the function $\{(0,3),(1,5),(2,7),(3,9)\}$. Determine the domain and range of the inverse function.

Solution:

This function is one-to-one since every x-value is paired with exactly one y-value.

To find the inverse we reverse the x-values and y-values in the ordered pairs of the function.

Function	$\{(0,3),(1,5),(2,7),(3,9)\}$
Inverse Function	$\{(3,0),(5,1),(7,2),(9,3)\}$
Domain of Inverse Function	$\{3,5,7,9\}$
Range of Inverse Function	$\{0,1,2,3\}$

Note:
Exercise:

Problem:

Find the inverse of $\{(0,4),(1,7),(2,10),(3,13)\}$. Determine the domain and range of the inverse function.

Solution:

Inverse function: $\{(4,0),(7,1),(10,2),(13,3)\}$. Domain: $\{4,7,10,13\}$. Range: $\{0,1,2,3\}$.

Note:
Exercise:

Problem:

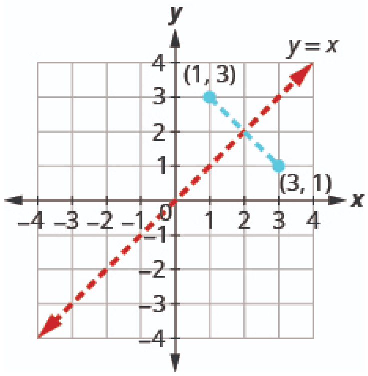
Find the inverse of $\{(-1,4),(-2,1),(-3,0),(-4,2)\}$. Determine the domain and range of the inverse function.

Solution:

Inverse function: $\{(4,-1),(1,-2),(0,-3),(2,-4)\}$. Domain: $\{0,1,2,4\}$. Range: $\{-4,-3,-2,-1\}$.

We just noted that if $f(x)$ is a one-to-one function whose ordered pairs are of the form (x,y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y,x).

So if a point (a,b) is on the graph of a function $f(x)$, then the ordered pair (b,a) is on the graph of $f^{-1}(x)$. See [link].



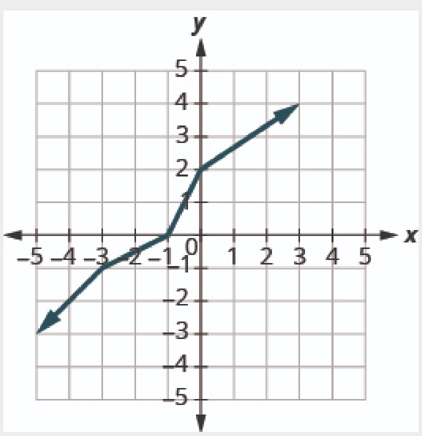
The distance between any two pairs (a, b) and (b, a) is cut in half by the line $y = x$. So we say the points are mirror images of each other through the line $y = x$.

Since every point on the graph of a function $f(x)$ is a mirror image of a point on the graph of $f^{-1}(x)$, we say the graphs are mirror images of each other through the line $y = x$. We will use this concept to graph the inverse of a function in the next example.

Example:
Exercise:

Problem:

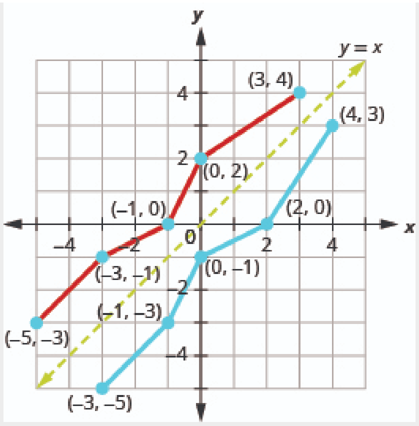
Graph, on the same coordinate system, the inverse of the one-to one function shown.



Solution:

We can use points on the graph to find points on the inverse graph. Some points on the graph are: $(-5, -3), (-3, -1), (-1, 0), (0, 2), (3, 4)$.

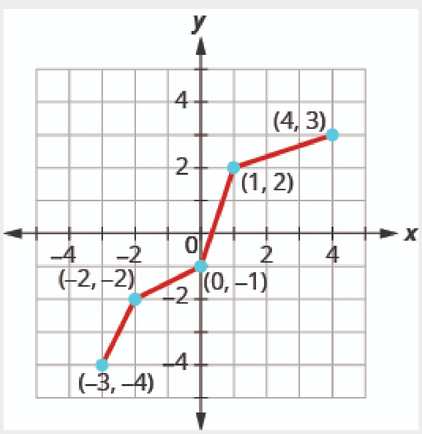
So, the inverse function will contain the points: $(-3, -5), (-1, -3), (0, -1), (2, 0), (4, 3)$.



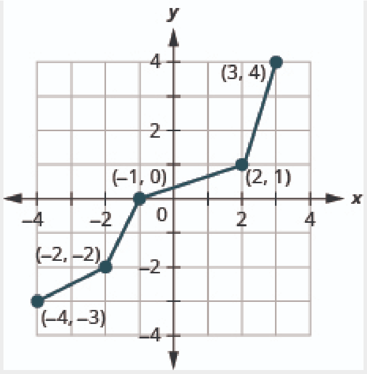
Notice how the graph of the original function and the graph of the inverse functions are mirror images through the line $y = x$.

Note:
Exercise:

Problem: Graph, on the same coordinate system, the inverse of the one-to one function.

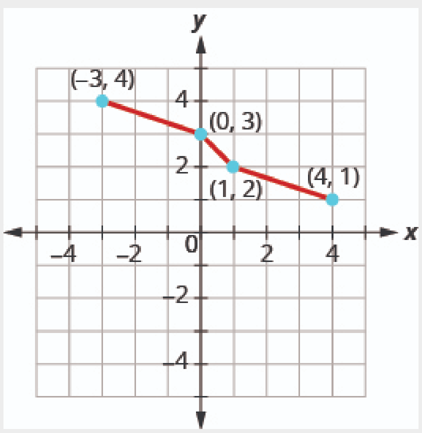


Solution:

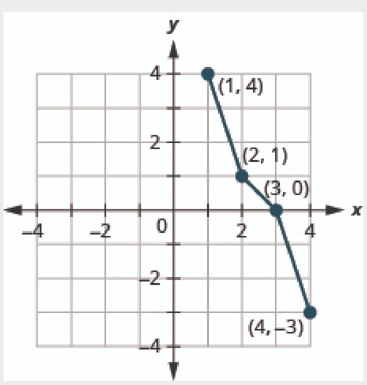


Note:
Exercise:

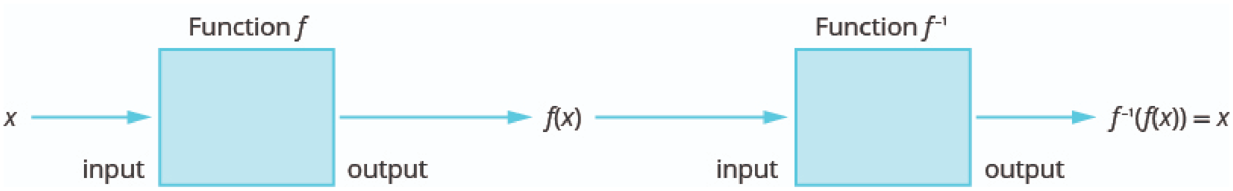
Problem: Graph, on the same coordinate system, the inverse of the one-to one function.



Solution:



When we began our discussion of an inverse function, we talked about how the inverse function ‘undoes’ what the original function did to a value in its domain in order to get back to the original *x*-value.



Note:
Inverse Functions
Equation:

$$\begin{array}{rcl} f^{-1}(f(x)) & = & x, \text{ for all } x \text{ in the domain of } f \\ f\left(f^{-1}(x)\right) & = & x, \text{ for all } x \text{ in the domain of } f^{-1} \end{array}$$

We can use this property to verify that two functions are inverses of each other.

Example:
Exercise:

Problem: Verify that $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions.

Solution:

The functions are inverses of each other if $g(f(x)) = x$ and $f(g(x)) = x$.

	$g(f(x)) \stackrel{?}{=} x$
Substitute $5x - 1$ for $f(x)$.	$g(5x - 1) \stackrel{?}{=} x$

Find $g(5x-1)$ where $g(x)=\frac{x+1}{5}$.	$\frac{(5x-1)+1}{5} \stackrel{?}{=} x$
Simplify.	$\frac{5x}{5} \stackrel{?}{=} x$
Simplify.	$x = x$ ✓
	$f(g(x)) \stackrel{?}{=} x$
Substitute $\frac{x+1}{5}$ for $g(x)$.	$f\left(\frac{x+1}{5}\right) \stackrel{?}{=} x$
Find $f\left(\frac{x+1}{5}\right)$ where $f(x)=5x-1$.	$5\left(\frac{x+1}{5}\right)-1 \stackrel{?}{=} x$
Simplify.	$x+1-1 \stackrel{?}{=} x$
Simplify.	$x = x$ ✓

Since both $g(f(x)) = x$ and $f(g(x)) = x$ are true, the functions $f(x) = 5x - 1$ and $g(x) = \frac{x+1}{5}$ are inverse functions. That is, they are inverses of each other.

Note:
Exercise:

Problem: Verify that the functions are inverse functions.

$f(x) = 4x - 3$ and $g(x) = \frac{x+3}{4}$.

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Note:
Exercise:

Problem: Verify that the functions are inverse functions.

$f\left(x\right)=2x+6$ and $g\left(x\right)=\frac{x-6}{2}$.

Solution:

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

We have found inverses of function defined by ordered pairs and from a graph. We will now look at how to find an inverse using an algebraic equation. The method uses the idea that if $f\left(x\right)$ is a one-to-one function with ordered pairs $\left(x,y\right)$, then its inverse function $f^{-1}\left(x\right)$ is the set of ordered pairs $\left(y,x\right)$.

If we reverse the *x* and *y* in the function and then solve for *y*, we get our inverse function.

Example:
How to Find the inverse of a One-to-One Function
Exercise:

Problem: Find the inverse of $f\left(x\right)=4x+7$.

Solution:

Step 1. Substitute *y* for *f*(*x*).	Replace *f*(*x*) with *y*.	$f(x)=4x+7$ $y=4x+7$
Step 2. Interchange the variables *x* and *y*.	Replace *x* with *y* and then *y* with *x*.	$x=4y+7$
Step 3. Solve for *y*.	Subtract 7 from each side. Divide by 4.	$x-7=4y$ $\frac{x-7}{4}=y$

Step 4. Substitute $f^{-1}(x)$ for y.	Replace y with $f^{-1}(x)$.	$\frac{x-7}{4} = f^{-1}(x)$
Step 5. Verify that the functions are inverses.	Show $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$	$f^{-1}(f(x)) \stackrel{?}{=} x$ $f^{-1}(4x+7) \stackrel{?}{=} x$ $\frac{(4x+7)-7}{4} \stackrel{?}{=} x$ $\frac{4x}{4} \stackrel{?}{=} x$ $x = x \checkmark$ $f(f^{-1}(x)) \stackrel{?}{=} x$ $f\left(\frac{x-7}{4}\right) \stackrel{?}{=} x$ $4\left(\frac{x-7}{4}\right) + 7 \stackrel{?}{=} x$ $x - 7 + 7 \stackrel{?}{=} x$ $x = x \checkmark$

Note:
Exercise:

Problem: Find the inverse of the function $f\left(x\right)=5x-3$.

Solution:

$f^{-1}(x) = \frac{x+3}{5}$

Note:
Exercise:

Problem: Find the inverse of the function $f\left(x\right)=8x+5$.

Solution:

$f^{-1}(x) = \frac{x-5}{8}$

We summarize the steps below.

Note:
How to Find the inverse of a One-to-One Function

Substitute y for $f\left(x\right)$.
Interchange the variables x and y.
Solve for y.
Substitute $f^{-1}\left(x\right)$ for y.
Verify that the functions are inverses.

Example:
How to Find the Inverse of a One-to-One Function
Exercise:

Problem: Find the inverse of $f\left(x\right)=\sqrt[5]{2x-3}$.

Solution:

$$\begin{array}{lrcl} & f\left(x\right) & = & \sqrt[5]{2x-3} \\ \text{Substitute } y \text{ for } f\left(x\right). & y & = & \sqrt[5]{2x-3} \\ \text{Interchange the variables } x \text{ and } y. & x & = & \sqrt[5]{2y-3} \\ \text{Solve for } y. & \left(x\right)^5 & = & \left(\sqrt[5]{2y-3}\right)^5 \\ & x^5 & = & 2y-3 \\ & x^5+3 & = & 2y \\ & \frac{x^5+3}{2} & = & y \\ \text{Substitute } f^{-1}\left(x\right) \text{ for } y. & f^{-1}\left(x\right) & = & \frac{x^5+3}{2} \end{array}$$

Verify that the functions are inverses.

$$\begin{aligned} f^{-1}(f(x)) &\stackrel{?}{=} x \\ f^{-1}\left(\sqrt[5]{2x-3}\right) &\stackrel{?}{=} x \\ \frac{\left(\sqrt[5]{2x-3}\right)^5+3}{2} &\stackrel{?}{=} x \\ \frac{2x-3+3}{2} &\stackrel{?}{=} x \\ \frac{2x}{2} &\stackrel{?}{=} x \\ x &= x\checkmark \end{aligned} \qquad \begin{aligned} f\left(f^{-1}(x)\right) &\stackrel{?}{=} x \\ f\left(\frac{x^5+3}{2}\right) &\stackrel{?}{=} x \\ \sqrt[5]{2\left(\frac{x^5+3}{2}\right)-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5+3-3} &\stackrel{?}{=} x \\ \sqrt[5]{x^5} &\stackrel{?}{=} x \\ x &= x\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = \sqrt[5]{3x-2}$.

Solution:

$f^{-1}(x) = \frac{x^5+2}{3}$

Note:
Exercise:

Problem: Find the inverse of the function $f(x) = \sqrt[4]{6x-7}$.

Solution:

$f^{-1}(x) = \frac{x^4+7}{6}$

Key Concepts

- **Composition of Functions:** The composition of functions f and g, is written $f \circ g$ and is defined by
  Equation:

$$(f \circ g)(x) = f(g(x))$$

We read $f(g(x))$ as f of g of x.

- **Horizontal Line Test:** If every horizontal line, intersects the graph of a function in at most one point, it is a one-to-one function.
- **Inverse of a Function Defined by Ordered Pairs:** If $f(x)$ is a one-to-one function whose ordered pairs are of the form (x, y), then its inverse function $f^{-1}(x)$ is the set of ordered pairs (y, x).
- **Inverse Functions:** For every x in the domain of one-to-one function f and f^{-1}, **Equation:**

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f\left(f^{-1}(x)\right) &= x \end{aligned}$$

- **How to Find the Inverse of a One-to-One Function:**

  Substitute y for $f(x)$.
  Interchange the variables x and y.
  Solve for y.
  Substitute $f^{-1}(x)$ for y.
  Verify that the functions are inverses.

Practice Makes Perfect

Find and Evaluate Composite Functions

In the following exercises, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.
Exercise:

Problem: $f(x) = 4x + 3$ and $g(x) = 2x + 5$

Solution:

ⓐ $8x + 23$ ⓑ $8x + 11$ ⓒ
$8x^2 + 26x + 15$

Exercise:

Problem: $f(x) = 3x - 1$ and $g(x) = 5x - 3$

Exercise:

Problem: $f(x) = 6x - 5$ and $g(x) = 4x + 1$

Solution:

ⓐ $24x+1$ ⓑ $24x-19$
ⓒ $24x^2+19x-5$

Exercise:

Problem: $f\left(x\right)=2x+7$ and $g\left(x\right)=3x-4$

Exercise:

Problem: $f\left(x\right)=3x$ and $g\left(x\right)=2x^2-3x$

Solution:

ⓐ $6x^2-9x$ ⓑ $18x^2-9x$
ⓒ $6x^3-9x^2$

Exercise:

Problem: $f\left(x\right)=2x$ and $g\left(x\right)=3x^2-1$

Exercise:

Problem: $f\left(x\right)=2x-1$ and $g\left(x\right)=x^2+2$

Solution:

ⓐ $2x^2+3$ ⓑ $4x^2-4x+3$
ⓒ $2x^3-x^2+4x-2$

Exercise:

Problem: $f\left(x\right)=4x+3$ and $g\left(x\right)=x^2-4$

In the following exercises, find the values described.

Exercise:

Problem: For functions $f\left(x\right)=2x^2+3$ and $g\left(x\right)=5x-1$, find
ⓐ $\left(f\circ g\right)\left(-2\right)$
ⓑ $\left(g\circ f\right)\left(-3\right)$
ⓒ $\left(f\circ f\right)\left(-1\right)$

Solution:

ⓐ 245 ⓑ 104 ⓒ 53

Exercise:

Problem: For functions $f(x) = 5x^2 - 1$ and $g(x) = 4x - 1$, find
ⓐ $(f \circ g)(1)$
ⓑ $(g \circ f)(-1)$
ⓒ $(f \circ f)(2)$

Exercise:

Problem: For functions $f(x) = 2x^3$ and $g(x) = 3x^2 + 2$, find
ⓐ $(f \circ g)(-1)$
ⓑ $(g \circ f)(1)$
ⓒ $(g \circ g)(1)$

Solution:

ⓐ 250 ⓑ 14 ⓒ 77

Exercise:

Problem: For functions $f(x) = 3x^3 + 1$ and $g(x) = 2x^2 - 3$, find
ⓐ $(f \circ g)(-2)$
ⓑ $(g \circ f)(-1)$
ⓒ $(g \circ g)(1)$

Determine Whether a Function is One-to-One

In the following exercises, determine if the set of ordered pairs represents a function and if so, is the function one-to-one.

Exercise:

Problem: $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

Solution:

Function; not one-to-one

Exercise:

Problem: $\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$

Exercise:

Problem: $\{(-3,-5),(-2,-3),(-1,-1),(0,1),(1,3),(2,5),(3,7)\}$

Solution:

One-to-one function

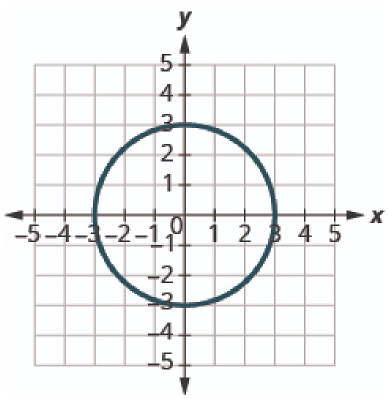
Exercise:

Problem: $\{(5,3),(4,2),(3,1),(2,0),$
$(1,-1),(0,-2),(-1,-3)\}$

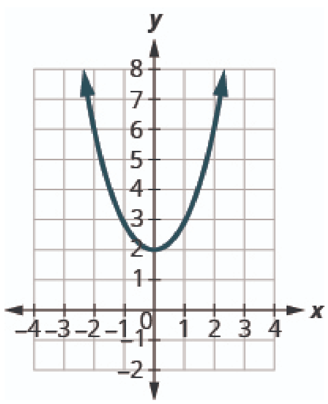
In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

Exercise:

Problem: ⓐ



ⓑ

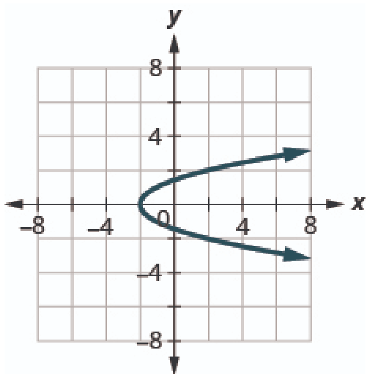


Solution:

ⓐ Not a function ⓑ Function; not one-to-one

Exercise:

Problem: ⓐ



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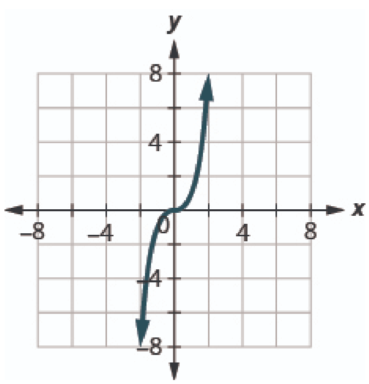
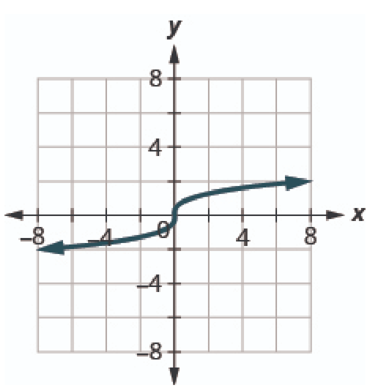
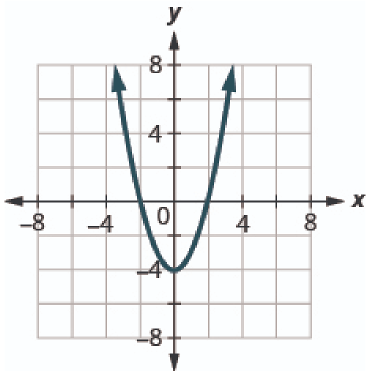


Exercise:

Problem: ⓐ



ⓑ

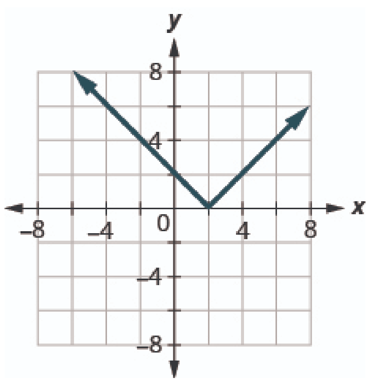


Solution:

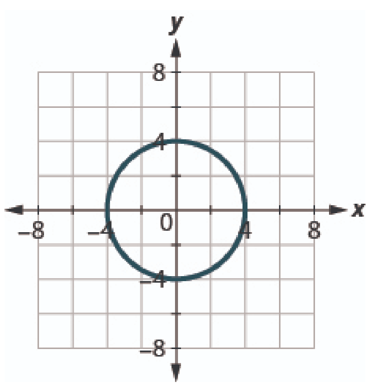
ⓐ One-to-one function
ⓑ Function; not one-to-one

Exercise:

Problem: ⓐ



ⓑ



In the following exercises, find the inverse of each function. Determine the domain and range of the inverse function.
Exercise:

Problem: $\{(2,1),(4,2),(6,3),(8,4)\}$

Solution:

Inverse function: $\{(1,2),(2,4),(3,6),(4,8)\}$. Domain: $\{1,2,3,4\}$. Range: $\{2,4,6,8\}$.

Exercise:

Problem: $\{(6,2),(9,5),(12,8),(15,11)\}$

Exercise:

Problem: $\{(0,-2),(1,3),(2,7),(3,12)\}$

Solution:

Inverse function: $\{(-2,0),(3,1),(7,2),(12,3)\}$. Domain: $\{-2,3,7,12\}$. Range: $\{0,1,2,3\}$.

Exercise:

Problem: $\{(0,0),(1,1),(2,4),(3,9)\}$

Exercise:

Problem: $\{(-2,-3),(-1,-1),(0,1),(1,3)\}$

Solution:

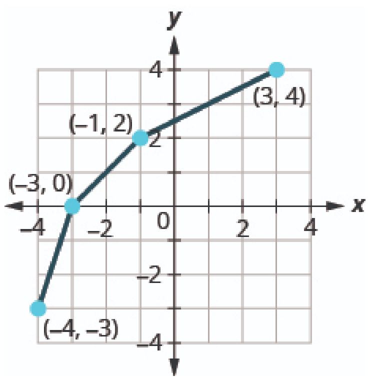
Inverse function: $\{(-3,-2),(-1,-1),(1,0),(3,1)\}$. Domain: $\{-3,-1,1,3\}$. Range: $\{-2,-1,0,1\}$.

Exercise:

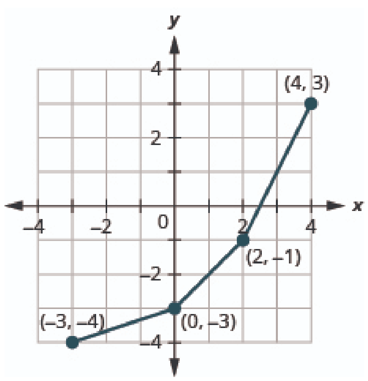
Problem: $\{(5,3),(4,2),(3,1),(2,0)\}$

In the following exercises, graph, on the same coordinate system, the inverse of the one-to-one function shown.
Exercise:

Problem:

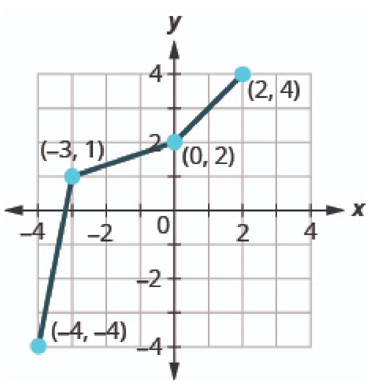


Solution:



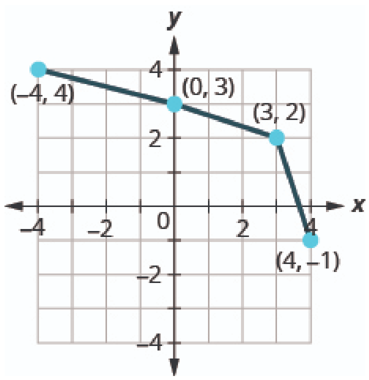
Exercise:

Problem:

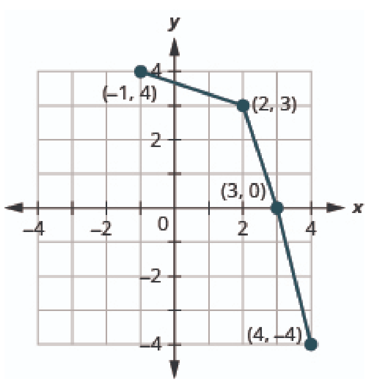


Exercise:

Problem:

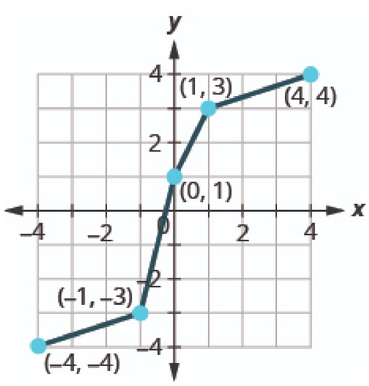


Solution:



Exercise:

Problem:



In the following exercises, determine whether or not the given functions are inverses.

Exercise:

Problem: $f\left(x\right)=x+8$ and $g\left(x\right)=x-8$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = x - 9$ and $g\left(x\right) = x + 9$

Exercise:

Problem: $f\left(x\right) = 7x$ and $g\left(x\right) = \frac{x}{7}$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = \frac{x}{11}$ and $g\left(x\right) = 11x$

Exercise:

Problem: $f\left(x\right) = 7x + 3$ and $g\left(x\right) = \frac{x-3}{7}$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses.

Exercise:

Problem: $f\left(x\right) = 5x - 4$ and $g\left(x\right) = \frac{x-4}{5}$

Exercise:

Problem: $f\left(x\right) = \sqrt{x+2}$ and $g\left(x\right) = x^2 - 2$

Solution:

$g(f(x)) = x$, and $f(g(x)) = x$, so they are inverses (for nonnegative x).

Exercise:

Problem: $f\left(x\right) = \sqrt[3]{x-4}$ and $g\left(x\right) = x^3 + 4$

In the following exercises, find the inverse of each function.

Exercise:

Problem: $f\left(x\right)=x-12$

Solution:

$f^{-1}(x)=x+12$

Exercise:

Problem: $f\left(x\right)=x+17$

Exercise:

Problem: $f\left(x\right)=9x$

Solution:

$f^{-1}(x)=\frac{x}{9}$

Exercise:

Problem: $f\left(x\right)=8x$

Exercise:

Problem: $f\left(x\right)=\frac{x}{6}$

Solution:

$f^{-1}(x)=6x$

Exercise:

Problem: $f\left(x\right)=\frac{x}{4}$

Exercise:

Problem: $f\left(x\right)=6x-7$

Solution:

$f^{-1}(x)=\frac{x+7}{6}$

Exercise:

Problem: $f\left(x\right)=7x-1$

Exercise:

Problem: $f(x) = -2x + 5$

Solution:

$f^{-1}(x) = \frac{x-5}{-2}$

Exercise:

Problem: $f(x) = -5x - 4$

Exercise:

Problem: $f(x) = x^2 + 6, x \geq 0$

Solution:

$f^{-1}(x) = \sqrt{x-6}$

Exercise:

Problem: $f(x) = x^2 - 9, x \geq 0$

Exercise:

Problem: $f(x) = x^3 - 4$

Solution:

$f^{-1}(x) = \sqrt[3]{x+4}$

Exercise:

Problem: $f(x) = x^3 + 6$

Exercise:

Problem: $f(x) = \frac{1}{x+2}$

Solution:

$f^{-1}(x) = \frac{1}{x} - 2$

Exercise:

Problem: $f(x) = \frac{1}{x-6}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x-2},x\ge 2$

Solution:

$f^{-1}(x)=x^2+2,x\ge 0$

Exercise:

Problem: $f\left(x\right)=\sqrt{x+8},x\ge -8$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{x-3}$

Solution:

$f^{-1}(x)=x^3+3$

Exercise:

Problem: $f\left(x\right)=\sqrt[3]{x+5}$

Exercise:

Problem: $f\left(x\right)=\sqrt[4]{9x-5},x\ge \frac{5}{9}$

Solution:

$f^{-1}(x)=\frac{x^4+5}{9},x\ge 0$

Exercise:

Problem: $f\left(x\right)=\sqrt[4]{8x-3},x\ge \frac{3}{8}$

Exercise:

Problem: $f\left(x\right)=\sqrt[5]{-3x+5}$

Solution:

$f^{-1}(x)=\frac{x^5-5}{-3}$

Exercise:

Problem: $f\left(x\right)=\sqrt[5]{-4x-3}$

Writing Exercises

Exercise:

Problem:

Explain how the graph of the inverse of a function is related to the graph of the function.

Solution:

Answers will vary.

Exercise:

Problem:

Explain how to find the inverse of a function from its equation. Use an example to demonstrate the steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find and evaluate composite functions.			
determine whether a function is one-to-one.			
find the inverse of a function.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

one-to-one function

A function is one-to-one if each value in the range has exactly one element in the domain. For each ordered pair in the function, each *y*-value is matched with only one *x*-value.

Evaluate and Graph Exponential Functions
By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
   If you missed this problem, review [link].
2. Evaluate: ⓐ 2^0 ⓑ $\left(\frac{1}{3}\right)^0$.
   If you missed this problem, review [link].
3. Evaluate: ⓐ 2^{-1} ⓑ $\left(\frac{1}{3}\right)^{-1}$.
   If you missed this problem, review [link].

Graph Exponential Functions

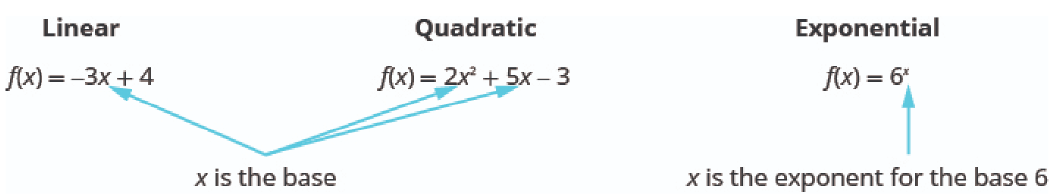
The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f\left(x\right)=a^x$ where $a>0$ and $a\neq 1$.

Note:
Exponential Function
An exponential function, where $a>0$ and $a\neq 1$, is a function of the form
Equation:

$$f\left(x\right)=a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.



Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

Equation:

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\tfrac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\tfrac{1}{2}\right) &= \sqrt{-4} \quad \text{not a real number} \end{aligned}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

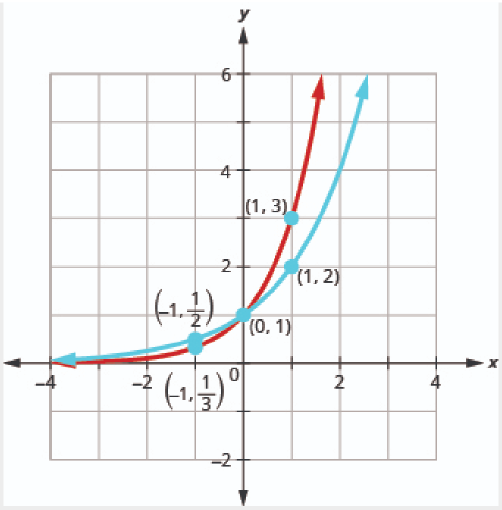
Example:
Exercise:

Problem: On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.

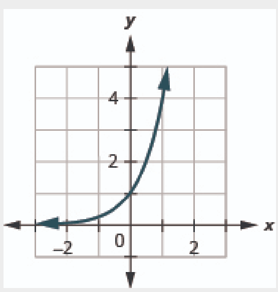
x	$f(x) = 2^x$	$(x, f(x))$		$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)		$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)		$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)		$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)		$3^3 = 27$	(3, 27)



Note:
Exercise:

Problem: Graph: $f\left(x\right)=4^{x}$.

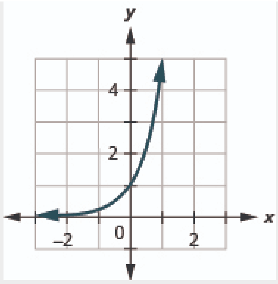
Solution:



Note:
Exercise:

Problem: Graph: $g\left(x\right)=5^{x}$.

Solution:



If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = 2^x$ contained $\left(-1, \frac{1}{2}\right)$ and the graph of $g(x) = 3^x$ contained $\left(-1, \frac{1}{3}\right)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

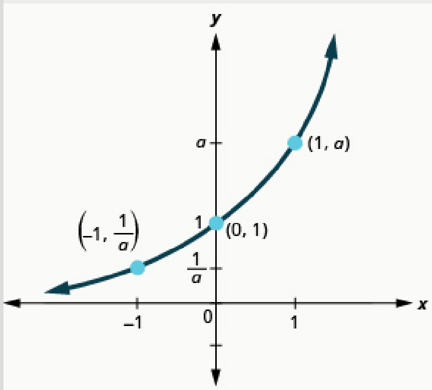
Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the x-axis, a horizontal asymptote.

Note:
Properties of the Graph of $f(x) = a^x$ when $a > 1$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$

x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$



Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$? The next example will explore this possibility.

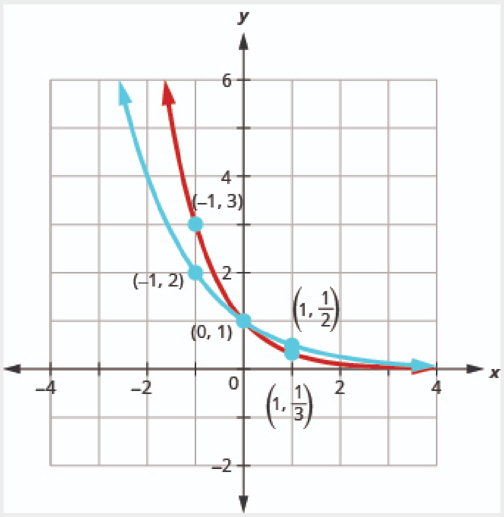
Example:
Exercise:

Problem: On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.

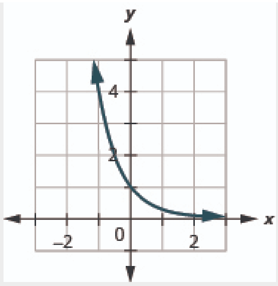
x	$f(x) = \left(\frac{1}{2}\right)^x$	$(x, f(x))$		$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$		$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$		$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0 = 1$	$(0, 1)$		$\left(\frac{1}{3}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$		$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$		$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$		$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$



Note:
Exercise:

Problem: Graph: $f(x) = \left(\frac{1}{4}\right)^x$.

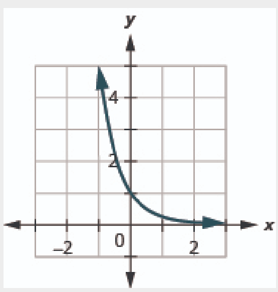
Solution:



Note:
Exercise:

Problem: Graph: $g\left(x\right)=\left(\frac{1}{5}\right)^{x}$.

Solution:



Now let’s look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<a<1$.

The graphs of $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$ and $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$ as well as the graphs of $f\left(x\right)=\left(\frac{1}{4}\right)^{x}$ and $g\left(x\right)=\left(\frac{1}{5}\right)^{x}$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^{0}=1$ for any a.

As before, the graph of each function, $f\left(x\right)=a^{x}$, also contains the point $\left(1,a\right)$. The graph of $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$ contained $\left(1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$ contained $\left(1,\frac{1}{3}\right)$. This makes sense as $a^{1}=a$.

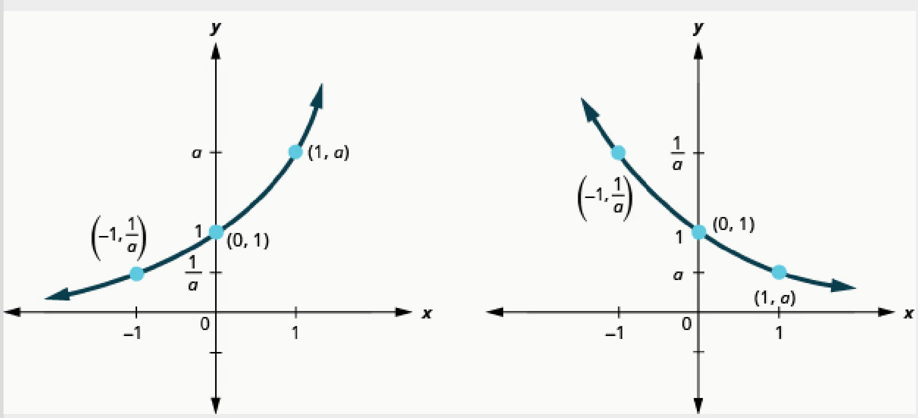
Notice too that the graph of each function, $f(x) = a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Note:
Properties of the Graph of $f(x) = a^x$

when $a > 1$		**when $0 < a < 1$**	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

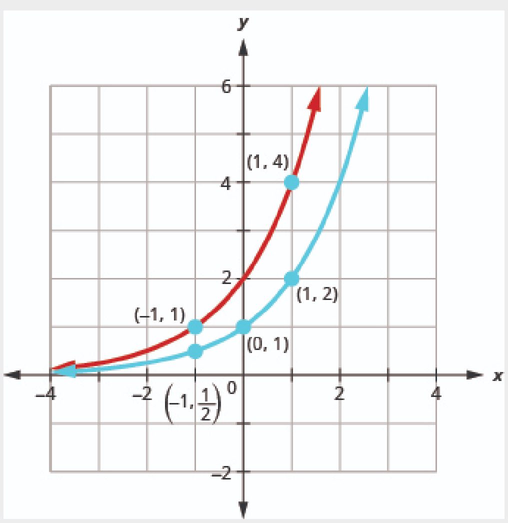
Example:
Exercise:

Problem: On the same coordinate system graph $f\left(x\right)=2^{x}$ and $g\left(x\right)=2^{x+1}$.

Solution:

We will use point plotting to graph the functions.

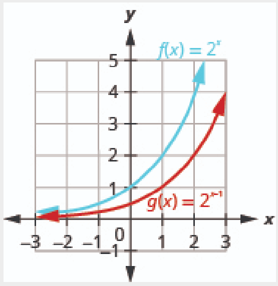
x	$f(x)=2^x$	$(x, f(x))$		$g(x)=2^{x+1}$	$(x, g(x))$
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$	$\left(-2,\frac{1}{4}\right)$		$2^{-2+1}=\frac{1}{2^1}=\frac{1}{2}$	$\left(-2,\frac{1}{2}\right)$
-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}$	$\left(-1,\frac{1}{2}\right)$		$2^{-1+1}=2^0=1$	$(-1, 1)$
0	$2^0=1$	$(0, 1)$		$2^{0+1}=2^1=2$	$(0, 2)$
1	$2^1=2$	$(1, 2)$		$2^{1+1}=2^2=4$	$(1, 4)$
2	$2^2=4$	$(2, 4)$		$2^{2+1}=2^3=8$	$(2, 8)$
3	$2^3=8$	$(3, 8)$		$2^{3+1}=2^4=16$	$(3, 16)$



Note:
Exercise:

Problem: On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

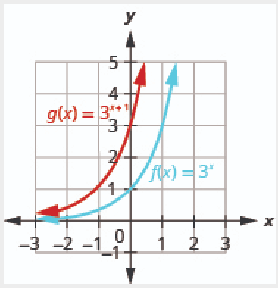
Solution:



Note:
Exercise:

Problem: On the same coordinate system, graph: $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Solution:



Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

Example:

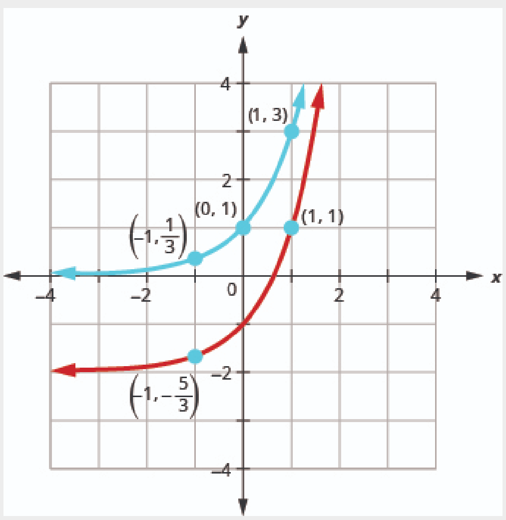
Exercise:

Problem: On the same coordinate system graph $f\left(x\right)={3}^{x}$ and $g\left(x\right)={3}^{x}-2.$

Solution:

We will use point plotting to graph the functions.

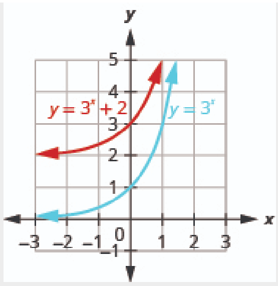
x	$f(x)=3^x$	$(x, g(x))$		$g(x)=3^x-2$	$(x, g(x))$
-2	$3^{-2}=\frac{1}{9}$	$\left(-2,\frac{1}{9}\right)$		$3^{-2}-2=\frac{1}{9}-2=-\frac{17}{9}$	$\left(-2,-\frac{17}{9}\right)$
-1	$3^{-1}=\frac{1}{3}$	$\left(-1,\frac{1}{3}\right)$		$3^{-1}-2=\frac{1}{3}-2=-\frac{5}{3}$	$\left(-1,-\frac{5}{3}\right)$
0	$3^0=1$	$(0, 1)$		$3^0-2=1-2=-1$	$(0, -1)$
1	$3^1=3$	$(1, 3)$		$3^1-2=3-2=1$	$(1, 1)$
2	$3^2=9$	$(2, 9)$		$3^2-2=9-2=7$	$(2, 8)$



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right)={3}^{x}$ and $g\left(x\right)={3}^{x}+2.$

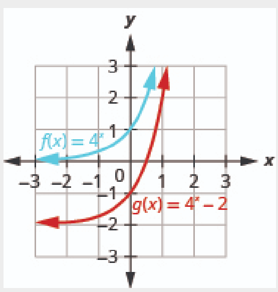
Solution:



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right) = 4^x$ and $g\left(x\right) = 4^x - 2$.

Solution:



Looking at the graphs of the functions $f\left(x\right) = 3^x$ and $g\left(x\right) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, *e*. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as *n* gets larger and larger. We say, as *n* approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^n$ for several values of n.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829…
1,000	2.716923932…
10,000	2.718145927…
100,000	2.718268237…
1,000,000	2.718280469…
1,000,000,000	2.718281827…

Equation:

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Note:
Natural Base e
The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,
Equation:

$$e \approx 2.718281827...$$

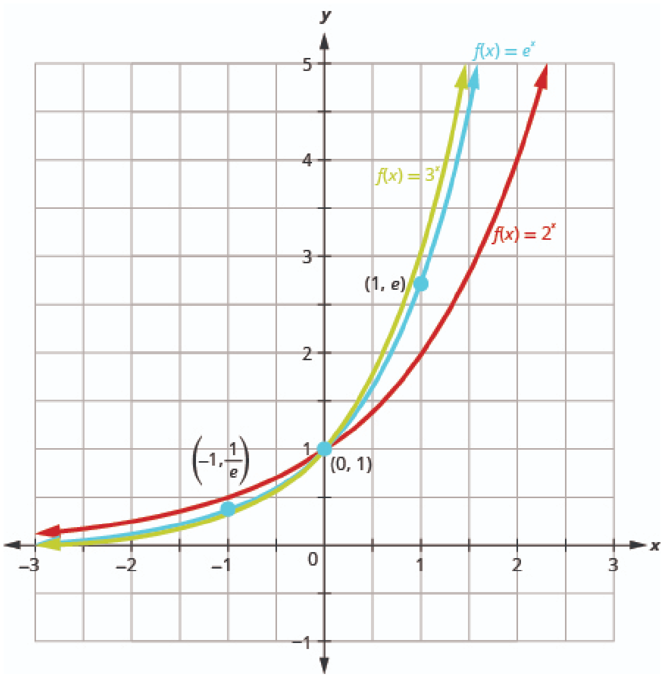
The exponential function whose base is e, $f\left(x\right)=e^x$ is called the **natural exponential function**.

Note:
Natural Exponential Function
The natural exponential function is an exponential function whose base is e
Equation:

$$f\left(x\right) = e^{x}$$

The domain is $\left(-\infty, \infty\right)$ and the range is $\left(0, \infty\right)$.

Let's graph the function $f\left(x\right) = e^{x}$ on the same coordinate system as $g\left(x\right) = 2^{x}$ and $h\left(x\right) = 3^{x}$.



Notice that the graph of $f\left(x\right) = e^{x}$ is "between" the graphs of $g\left(x\right) = 2^{x}$ and $h\left(x\right) = 3^{x}$. Does this make sense as $2 < e < 3$?

Solve Exponential Equations

Equations that include an exponential expression a^{x} are called exponential equations. To solve them we use a property that says as long as $a > 0$ and $a \neq 1$, if $a^{x} = a^{y}$ then it is true that $x = y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal.

Note:
One-to-One Property of Exponential Equations
For $a > 0$ and $a \neq 1$,
Equation:

$$\text{If } a^{x} = a^{y}, \text{ then } x = y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Example:
How to Solve an Exponential Equation
Exercise:

Problem: Solve: $3^{2x-5}=27$.

Solution:

Step 1. Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27=3^3$	$3^{2x-5}=27$ $3^{2x-5}=3^3$
Step 2. Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x-5=3$
Step 3. Solve the equation.	Add 5 to each side. Divide by 2.	$2x=8$ $x=4$
Step 4. Check the solution.	Substitute $x=4$ into the original equation.	$3^{2x-5}=27$ $3^{2\cdot4-5}\stackrel{?}{=}27$ $3^3\stackrel{?}{=}27$ $27=27$ ✓

Note:
Exercise:

Problem: Solve: $3^{3x-2}=81$.

Solution:

$x=2$

Note:
Exercise:

Problem: Solve: $7^{x-3}=7$.

Solution:

$x = 4$

The steps are summarized below.

Note:
How to Solve an Exponential Equation

Write both sides of the equation with the same base, if possible.
Write a new equation by setting the exponents equal.
Solve the equation.
Check the solution.

In the next example, we will use our properties on exponents.

Example:
Exercise:

Problem: Solve $\frac{e^{x^2}}{e^3} = e^{2x}$.

Solution:

	$\frac{e^{x^2}}{e^3} = e^{2x}$
Use the Property of Exponents: $\frac{a^m}{a^n} = a^{m-n}$.	$e^{x^2-3} = e^{2x}$
Write a new equation by setting the exponents equal.	$x^2 - 3 = 2x$
Solve the equation.	$x^2 - 2x - 3 = 0$
	$(x-3)(x+1) = 0$
	$x = 3,\ \ x = -1$
Check the solutions.	

$x = 3$	$x = -1$
$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$	$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$
$\frac{e^{3^2}}{e^3} \stackrel{?}{=} e^{2 \cdot 3}$	$\frac{e^{(-1)^2}}{e^3} \stackrel{?}{=} e^{2 \cdot (-1)}$
$\frac{e^9}{e^3} \stackrel{?}{=} e^6$	$\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$
$e^6 = e^6$ ✓	$e^{-2} = e^{-2}$ ✓

Note:
Exercise:

Problem: Solve: $\frac{e^{x^2}}{e^x} = e^2$.

Solution:

$x = -1, x = 2$

Note:
Exercise:

Problem: Solve: $\frac{e^{x^2}}{e^x} = e^6$.

Solution:

$x = -2, x = 3$

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, *P*, is invested at an interest rate, *r*, for *t* years, the new balance, *A*, will depend on how often the interest is compounded. If the interest is compounded *n* times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Note:

Compound Interest
For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:
Equation:

$$\begin{array}{ll} A = P\left(1+\frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A = Pe^{rt} & \text{when compounded continuously.} \end{array}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example:
Exercise:

Problem:

A total of $10,000 was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10{,}000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ
For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1+\frac{0.05}{4}\right)^{4\cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24{,}459.20$$

ⓑ
For monthly compounding, $n = 12$. There are 12 months in a year. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Substitute the values in the formula. $A = 10{,}000\left(1 + \frac{0.05}{12}\right)^{12\cdot18}$

Compute the amount. $A = \$24{,}550.08$

ⓒ
For compounding continuously, $A = Pe^{rt}$

Substitute the values in the formula. $A = 10{,}000e^{0.05\cdot18}$

Compute the amount. $A = \$24{,}596.03$

Note:
Exercise:

Problem:

Angela invested \$15,000 in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ \$22,332.96
ⓑ \$22,362.49 ⓒ \$22,377.37

Note:
Exercise:

Problem:

Allan invested \$10,000 in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ \$21,071.81 ⓑ \$21,137.04
ⓒ \$21,170.00

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Note:
Exponential Growth and Decay
For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:
Equation:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example:
Exercise:

Problem:

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas.	$A = ?$
Be sure to put the percent in decimal form.	$A_0 = 100$
Be sure the units match—the rate is per hour and	$r = 0.25/\text{hour}$
the time is in hours.	$t = 24\text{ hours}$
Substitute the values in the formula: $A = A_0 e^{rt}$.	$A = 100e^{0.25\cdot 24}$
Compute the amount.	$A = 40{,}342.88$
Round to the nearest whole virus.	$A = 40{,}343$
	The researcher will find 40,343 viruses.

Note:
Exercise:

Problem:

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

She will find 166 bacteria.

Note:
Exercise:

Problem:

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Solution:

She will find 1,102 viruses.

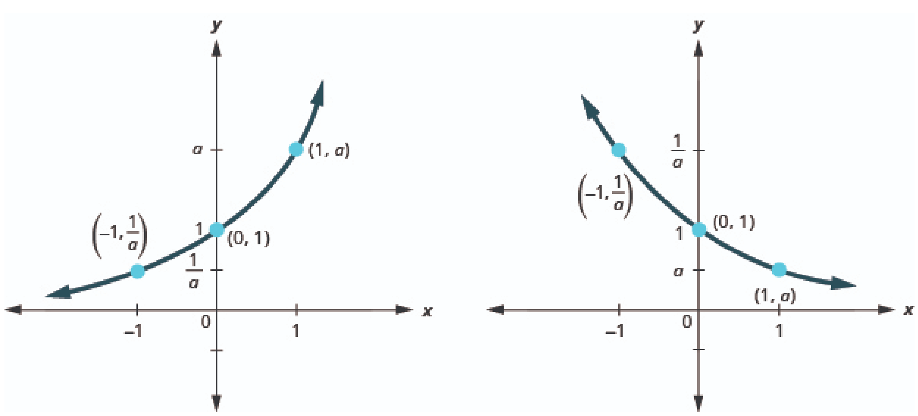
Note:
Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions
- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$	Contains	$(1, a),\ \left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,
  Equation:

$$A = A_0 e^{rt}$$

- **How to Solve an Exponential Equation**

  Write both sides of the equation with the same base, if possible.
  Write a new equation by setting the exponents equal.
  Solve the equation.
  Check the solution.

- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	when compounded n times a year.
$A = Pe^{rt}$	when compounded continuously.

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount,A, is $A = A_0e^{rt}$.

Practice Makes Perfect

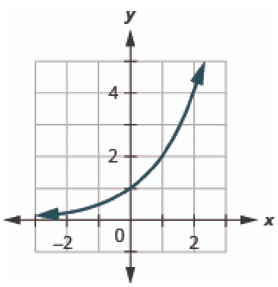
Graph Exponential Functions

In the following exercises, graph each exponential function.

Exercise:

Problem: $f\left(x\right) = 2^x$

Solution:



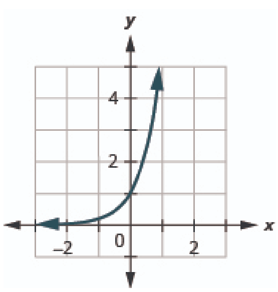
Exercise:

Problem: $g\left(x\right) = 3^x$

Exercise:

Problem: $f\left(x\right) = 6^x$

Solution:



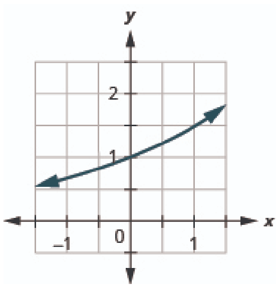
Exercise:

Problem: $g\left(x\right) = 7^x$

Exercise:

Problem: $f\left(x\right)=\left(1.5\right)^{x}$

Solution:



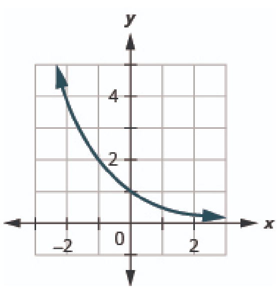
Exercise:

Problem: $g\left(x\right)=\left(2.5\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$

Solution:



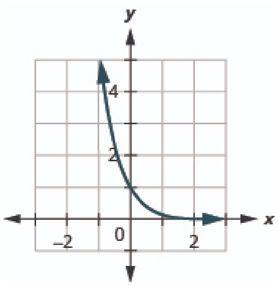
Exercise:

Problem: $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{6}\right)^{x}$

Solution:



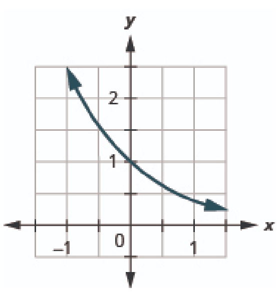
Exercise:

Problem: $g\left(x\right)=\left(\frac{1}{7}\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(0.4\right)^{x}$

Solution:



Exercise:

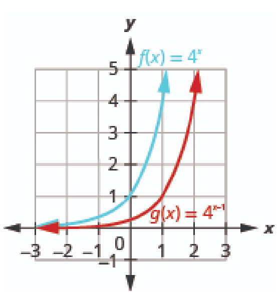
Problem: $g\left(x\right)=\left(0.6\right)^{x}$

In the following exercises, graph each function in the same coordinate system.

Exercise:

Problem: $f\left(x\right)=4^{x},\ g\left(x\right)=4^{x-1}$

Solution:



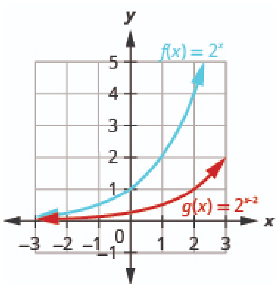
Exercise:

Problem: $f\left(x\right)=3^{x},\ g\left(x\right)=3^{x-1}$

Exercise:

Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x-2}$

Solution:



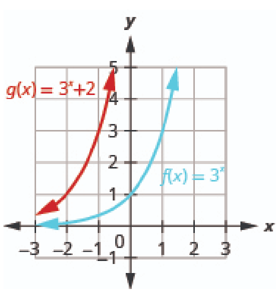
Exercise:

Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x+2}$

Exercise:

Problem: $f\left(x\right)=3^{x},\ g\left(x\right)=3^{x}+2$

Solution:



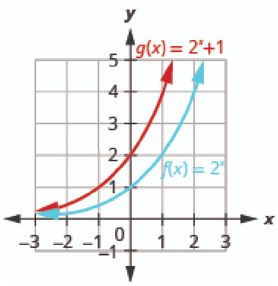
Exercise:

Problem: $f\left(x\right)=4^{x},\ g\left(x\right)=4^{x}+2$

Exercise:

Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x}+1$

Solution:



Exercise:

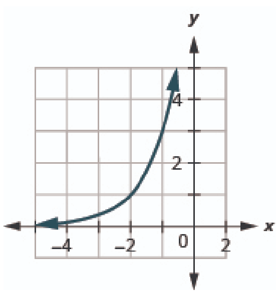
Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x}-1$

In the following exercises, graph each exponential function.

Exercise:

Problem: $f\left(x\right)=3^{x+2}$

Solution:



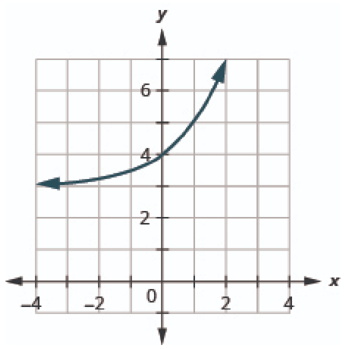
Exercise:

Problem: $f\left(x\right)=3^{x-2}$

Exercise:

Problem: $f\left(x\right)=2^{x}+3$

Solution:



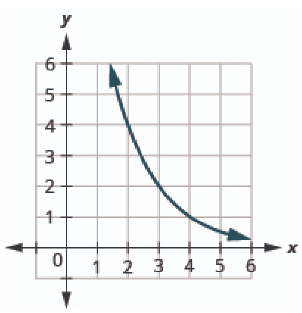
Exercise:

Problem: $f(x) = 2^x - 3$

Exercise:

Problem: $f(x) = \left(\frac{1}{2}\right)^{x-4}$

Solution:



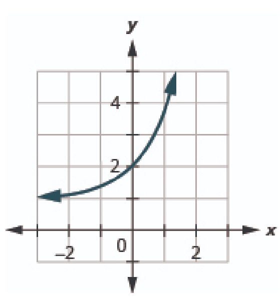
Exercise:

Problem: $f(x) = \left(\frac{1}{2}\right)^x - 3$

Exercise:

Problem: $f(x) = e^x + 1$

Solution:



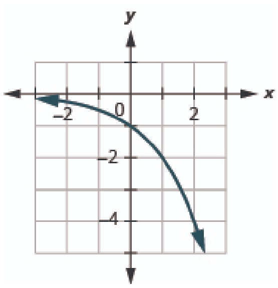
Exercise:

Problem: $f\left(x\right)=e^{x-2}$

Exercise:

Problem: $f\left(x\right)=-2^{x}$

Solution:



Exercise:

Problem: $f\left(x\right)=3^{x}$

Solve Exponential Equations

In the following exercises, solve each equation.

Exercise:

Problem: $2^{3x-8}=16$

Solution:

$x=4$

Exercise:

Problem: $2^{2x-3}=32$

Exercise:

Problem: $3^{x+3}=9$

Solution:

$x=-1$

Exercise:

Problem: $3^{x^{2}}=81$

Exercise:

Problem: $4^{x^2} = 4$

Solution:

$x = -1, x = 1$

Exercise:

Problem: $4^x = 32$

Exercise:

Problem: $4^{x+2} = 64$

Solution:

$x = 1$

Exercise:

Problem: $4^{x+3} = 16$

Exercise:

Problem: $2^{x^2+2x} = \frac{1}{2}$

Solution:

$x = -1$

Exercise:

Problem: $3^{x^2-2x} = \frac{1}{3}$

Exercise:

Problem: $e^{3x} \cdot e^4 = e^{10}$

Solution:

$x = 2$

Exercise:

Problem: $e^{2x} \cdot e^3 = e^9$

Exercise:

Problem: $\frac{e^{x^2}}{e^2} = e^x$

Solution:

$x = -1, x = 2$

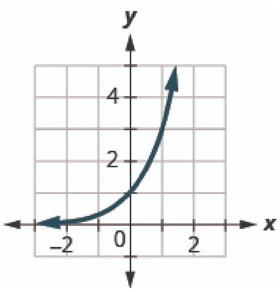
Exercise:

Problem: $\frac{e^{x^2}}{e^3} = e^{2x}$

In the following exercises, match the graphs to one of the following functions: ⓐ 2^x ⓑ 2^{x+1} ⓒ 2^{x-1} ⓓ $2^x + 2$ ⓔ $2^x - 2$ ⓕ 3^x

Exercise:

Problem:

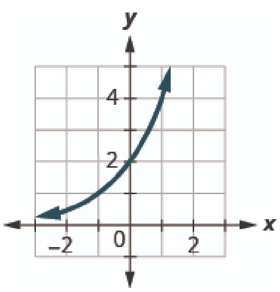


Solution:

ⓕ

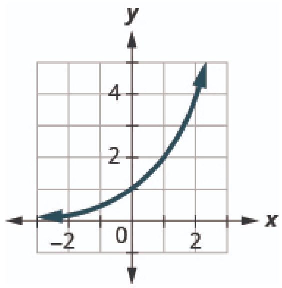
Exercise:

Problem:



Exercise:

Problem:

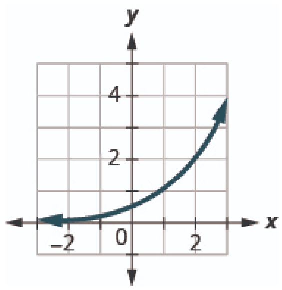


Solution:

ⓐ

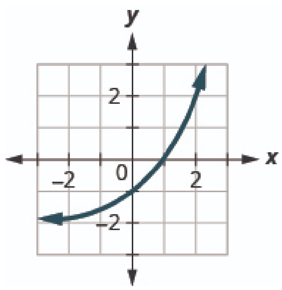
Exercise:

Problem:



Exercise:

Problem:

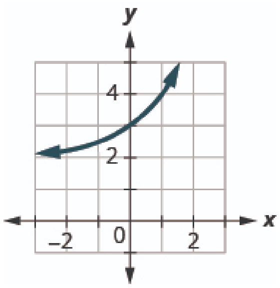


Solution:

ⓔ

Exercise:

Problem:



Use exponential models in applications

In the following exercises, use an exponential model to solve.

Exercise:

Problem:

Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Solution:

ⓐ $\$7,387.28$ ⓑ $\$7,434.57$ ⓒ $\$7,459.12$

Exercise:

Problem:

Cynthia invested $\$12,000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Exercise:

Problem:

Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

Solution:

$\$36,945.28$

Exercise:

Problem:

Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

Exercise:

Problem:

A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

223 bacteria

Exercise:

Problem:

A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Exercise:

Problem:

In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

Solution:

288,929,825

Exercise:

Problem:

In the last ten years the population of Brazil has grown at a rate of 0.9% per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

Writing Exercises

Exercise:

Problem:

Explain how you can distinguish between exponential functions and polynomial functions.

Solution:

Answers will vary.

Exercise:

Problem: Compare and contrast the graphs of $y = x^2$ and $y = 2^x$.

Exercise:

Problem:

What happens to an exponential function as the values of x decreases? Will the graph ever cross the
y-axis? Explain.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph exponential eunctions.			
solve exponential equations.			
use exponential models in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

asymptote
: A line which a graph of a function approaches closely but never touches.

exponential function
: An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

natural base
: The number *e* is defined as the value of $\left(1 + \frac{1}{n}\right)^n$, as *n* gets larger and larger. We say, as *n* increases without bound, $e \approx 2.718281827...$

natural exponential function
: The natural exponential function is an exponential function whose base is *e*: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Evaluate and Graph Logarithmic Functions
By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph Logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

Note:
Before you get started, take this readiness quiz.

1. Solve: $x^2 = 81$.
   If you missed this problem, review [link].
2. Evaluate: 3^{-2}.
   If you missed this problem, review [link].
3. Solve: $2^4 = 3x - 5$.
   If you missed this problem, review [link].

We have spent some time finding the inverse of many functions. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

	$f(x) = a^x$
Rewrite with $y = f(x)$.	$y = a^x$
Interchange the variables x and y.	$x = a^y$
Solve for y.	Oops! We have no way to solve for y!

To deal with this we define the logarithm function with base a to be the inverse of the exponential function $f(x) = a^x$. We use the notation $f^{-1}(x) = \log_a x$ and say the inverse function of the exponential function is the logarithmic function.

Note:
Logarithmic Function
The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0, x > 0$, and $a \neq 1$.
Equation:

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let’s take a close look at the equations. See [link]. Notice the positions of the exponent and base.

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, “base to the exponent give us the number.”

Example:
Exercise:

Problem: Convert to logarithmic form: ⓐ $2^3 = 8$, ⓑ $5^{\frac{1}{2}} = \sqrt{5}$, and ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{16}$.

Solution:

Identify the base and the exponent.

(a)

$2^3 = 8$

$y = \log_a x$

$3 = \log_2 8$

If $2^3 = 8$, then $3 = \log_2 8$.

(b)

$5^{\frac{1}{2}} = \sqrt{5}$

$y = \log_a x$

$\frac{1}{2} = \log_5 \sqrt{5}$

If $5^{\frac{1}{2}} = \sqrt{5}$, then $\frac{1}{2} = \log_5 \sqrt{5}$.

(c)

$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$y = \log_a x$

$4 = \log_{\frac{1}{2}} \frac{1}{16}$

If $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, then $4 = \log_{\frac{1}{2}} \frac{1}{16}$.

Note:
Exercise:

Problem: Convert to logarithmic form: ⓐ $3^2 = 9$ ⓑ $7^{\frac{1}{2}} = \sqrt{7}$ ⓒ $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Solution:

ⓐ $\log_3 9 = 2$
ⓑ $\log_7 \sqrt{7} = \frac{1}{2}$ ⓒ $\log_{\frac{1}{3}} \frac{1}{27} = x$

Note:
Exercise:

Problem: Convert to logarithmic form: ⓐ $4^3 = 64$ ⓑ $4^{\frac{1}{3}} = \sqrt[3]{4}$ ⓒ $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Solution:

ⓐ $\log_4 64 = 3$
ⓑ $\log_4 \sqrt[3]{4} = \frac{1}{3}$ ⓒ $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example:
Exercise:

Problem: Convert to exponential form: ⓐ $2 = \log_8 64$, ⓑ $0 = \log_4 1$, and ⓒ $-3 = \log_{10} \frac{1}{1000}$.

Solution:

Identify the base and the exponent.

(a)	(b)	(c)
$2 = \log_8 64$	$0 = \log_4 1$	$-3 = \log_{10} \frac{1}{1000}$
$x = a^y$	$x = a^y$	$x = a^y$
$64 = 8^2$	$1 = 4^0$	$\frac{1}{1000} = \log^{-3}$
If $2 = \log_8 64$, then $64 = 8^2$.	If $0 = \log_4 1$, then $1 = 4^0$.	If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$.

Note:
Exercise:

Problem: Convert to exponential form: ⓐ $3 = \log_4 64$ ⓑ $0 = \log_x 1$ ⓒ $-2 = \log_{10} \frac{1}{100}$

Solution:

ⓐ $64 = 4^3$
ⓑ $1 = x^0$ ⓒ $\frac{1}{100} = 10^{-2}$

Note:
Exercise:

Problem: Convert to exponential form: ⓐ $3 = \log_3 27$ ⓑ $0 = \log_x 1$ ⓒ $-1 = \log_{10}\frac{1}{10}$

Solution:

ⓐ $27 = 3^3$ ⓑ $1 = x^0$
ⓒ $\frac{1}{10} = 10^{-1}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example:
Exercise:

Problem: Find the value of *x*: ⓐ $\log_x 36 = 2$, ⓑ $\log_4 x = 3$, and ⓒ $\log_{\frac{1}{2}}\frac{1}{8} = x$.

Solution:

ⓐ

	$\log_x 36 = 2$	
Convert to exponential form.	$x^2 = 36$	
Solve the quadratic.	$x = 6, \quad \cancel{x = -6}$	
The base of a logarithmic function must be positive, so we eliminate $x = -6$.	$x = 6$	Therefore, $\log_6 36 = 2$.

ⓑ

	$\log_4 x = 3$	
Convert to exponential form.	$4^3 = x$	
Simplify.	$x = 64$	Therefore, $\log_4 64 = 3$.

ⓒ

	$\log_{\frac{1}{2}}\frac{1}{8} = x$	
Convert to exponential form.	$\left(\frac{1}{2}\right)^x = \frac{1}{8}$	
Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.	$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$	
With the same base, the exponents must be equal.	$x = 3$	Therefore, $\log_{\frac{1}{2}}\frac{1}{8} = 3$

Note:
Exercise:

Problem: Find the value of x : ⓐ $\log_x 64 = 2$ ⓑ $\log_5 x = 3$ ⓒ $\log_{\frac{1}{2}}\frac{1}{4} = x$

Solution:

ⓐ $x=8$ ⓑ $x=125$ ⓒ $x=2$

Note:
Exercise:

Problem: Find the value of x : ⓐ $\log_x 81=2$ ⓑ $\log_3 x=5$ ⓒ $\log_{\frac{1}{3}}\frac{1}{27}=x$

Solution:

ⓐ
$x=9$ ⓑ $x=243$ ⓒ $x=3$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means \\3 to what power will be 27//? Since $3^3=27$, we know $\log_3 27=3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_5 25$,
ⓑ $\log_9 3$, and ⓒ $\log_2\frac{1}{16}$.

Solution:

ⓐ

	$\log_5 25$	
5 to what power will be 25?	$\log_5 25 = 2$	
Or		
Set the expression equal to x.	$\log_5 25 = x$	
Change to exponential form.	$5^x = 25$	
Rewrite 25 as 5^2.	$5^x = 5^2$	
With the same base the exponents must be equal.	$x = 2$	Therefore, $\log_5 25=2$.

ⓑ

$$\begin{array}{lrcl} & \log_9 3 & & \\ \text{Set the expression equal to } x. & \log_9 3 & = & x \\ \text{Change to exponential form.} & 9^x & = & 3 \\ \text{Rewrite 9 as } 3^2. & \left(3^2\right)^x & = & 3^1 \\ \text{Simplify the exponents.} & 3^{2x} & = & 3^1 \\ \text{With the same base the exponents must be equal.} & 2x & = & 1 \\ \text{Solve the equation.} & x & = & \frac{1}{2} \quad \text{Therefore, } \log_9 3 = \frac{1}{2}. \end{array}$$

ⓒ

$$\begin{array}{lrcl} & \log_2 \frac{1}{16} & & \\ \text{Set the expression equal to } x. & \log_2 \frac{1}{16} & = & x \\ \text{Change to exponential form.} & 2^x & = & \frac{1}{16} \\ \text{Rewrite 16 as } 2^4. & 2^x & = & \frac{1}{2^4} \\ & 2^x & = & 2^{-4} \\ \text{With the same base the exponents must be equal.} & x & = & -4 \quad \text{Therefore, } \log_2 \frac{1}{16} = -4. \end{array}$$

Note:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_{12} 144$
ⓑ $\log_4 2$
ⓒ $\log_2 \frac{1}{32}$

Solution:

ⓐ
2 ⓑ $\frac{1}{2}$ ⓒ -5

Note:
Exercise:

Problem: Find the exact value of each logarithm without using a calculator:
ⓐ $\log_9 81$
ⓑ $\log_8 2$
ⓒ $\log_3 \frac{1}{9}$

Solution:

ⓐ 2 ⓑ $\frac{1}{3}$ ⓒ -2

Graph Logarithmic Functions

To graph a logarithmic function $y = \log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an *x*-value and then determine its corresponding *y*-value. In this case you may find it easier to choose *y*-values and then determine its corresponding *x*-value.

Example:
Exercise:

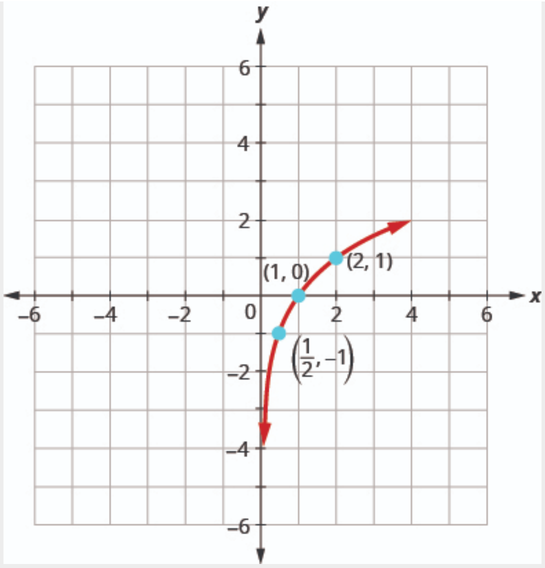
Problem: Graph $y = \log_2 x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

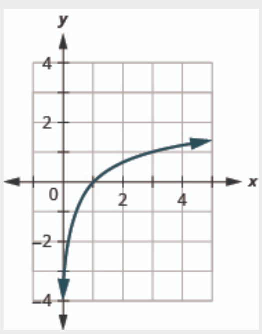
y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, 2\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$



Note:
Exercise:

Problem: Graph: $y = \log_3 x$.

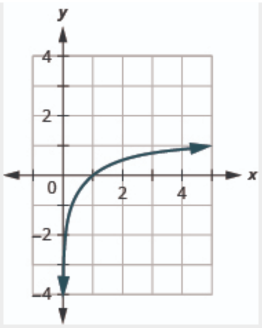
Solution:



Note:
Exercise:

Problem: Graph: $y = \log_5 x$.

Solution:



The graphs of $y = \log_2 x$,$y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = \log_a 1$ means $a^0 = 1$ which is true for any a.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any a.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the *y*-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

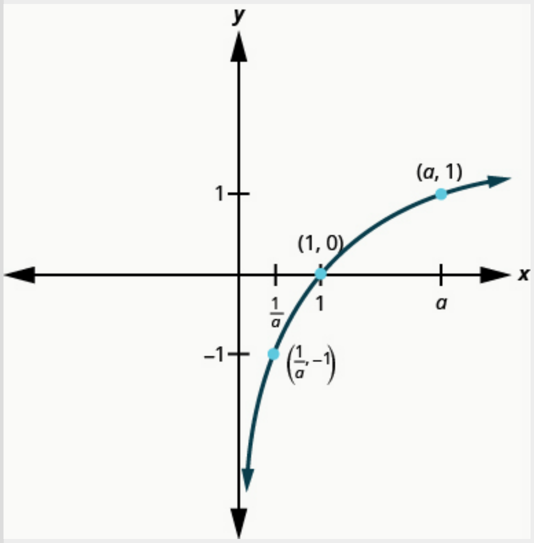
What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the *y*-axis so very closely but will never cross it, we call the line $x = 0$, the *y*-axis, a vertical asymptote.

Note:
Properties of the Graph of $y = \log_a x$ when $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$

y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y-axis



Our next example looks at the graph of $y = \log_a x$ when $0 < a < 1$.

Example:
Exercise:

Problem: Graph $y = \log_{\frac{1}{3}} x$.

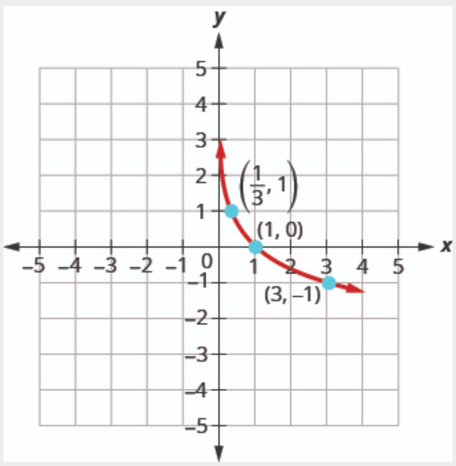
Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of *y* and then get *x*.

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$

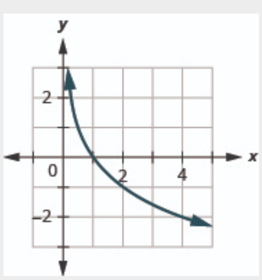
y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



Note:
Exercise:

Problem: Graph: $y = \log_{\frac{1}{2}} x$.

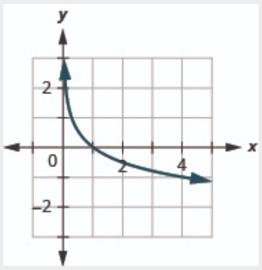
Solution:



Note:
Exercise:

Problem: Graph: $y = \log_{\frac{1}{4}} x$.

Solution:



Now, let's look at the graphs $y = \log_{\frac{1}{2}} x,\ y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points,$(1, 0)$,$(a, 1)$,$\left(\frac{1}{a}, -1\right)$. This make sense for the same reasons we argued above.

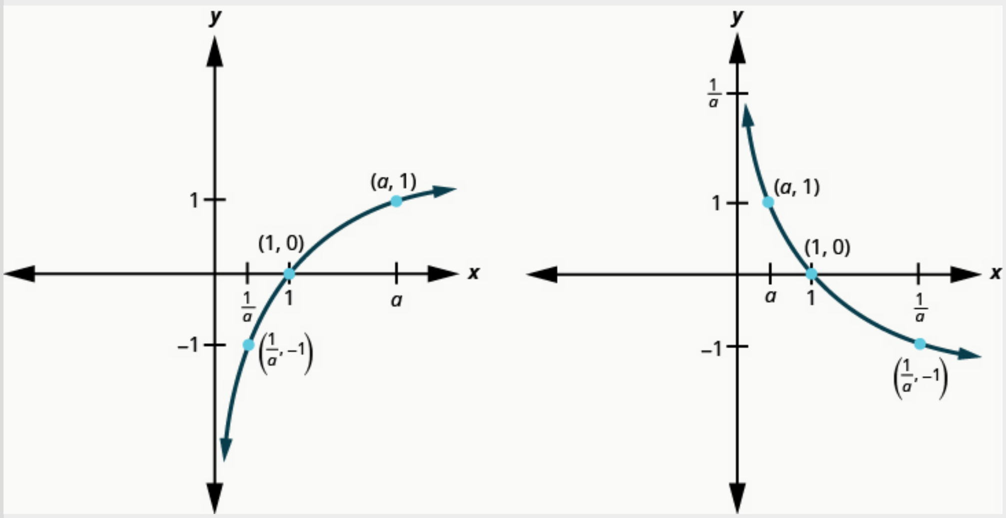
We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

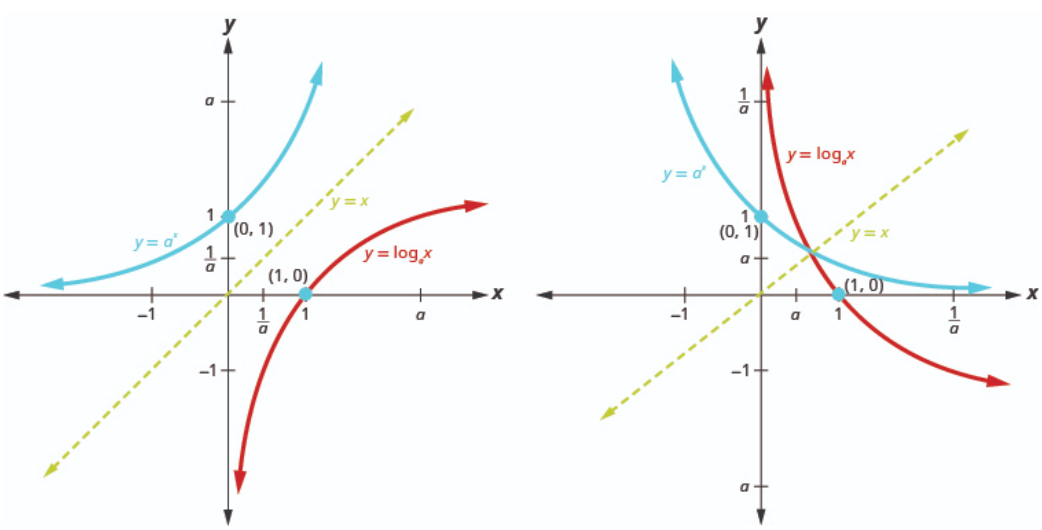
Note:
Properties of the Graph of $y = \log_a x$

when $a > 1$		**when** $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$

when $a > 1$		when $0 < a < 1$	
y-intercept	none	y-intercept	None
Contains	$(a, 1)$,$\left(\frac{1}{a}, -1\right)$	Contains	$(a, 1)$,$\left(\frac{1}{a}, -1\right)$
Asymptote	y-axis	Asymptote	y-axis
Basic shape	increasing	Basic shape	Decreasing



We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in [link] show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.



Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the *x*-axis is the horizontal asymptote for the exponential functions and the *y*-axis is the vertical asymptote for the logarithmic functions.

Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number *e*. Just as *e* was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base *e* is called the **natural logarithmic function**. The function $f\left(x\right) = \log_e x$ is generally written $f\left(x\right) = \ln x$ and we read it as "el en of x."

Note:
Natural Logarithmic Function
The function $f\left(x\right) = \ln x$ is the **natural logarithmic function** with base e, where $x > 0$.
Equation:

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is 10, we call it the **common logarithmic function** and the base is not shown. If the base *a* of a logarithm is not shown, we assume it is 10.

Note:
Common Logarithmic Function
The function $f\left(x\right) = \log x$ is the **common logarithmic function** with base10, where $x > 0$.
Equation:

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations, $y = \log_a x$, we need to remember that for the base *a*, $a > 0$ and $a \neq 1$. Also, the domain is $x > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

Example:
Exercise:

Problem: Solve: ⓐ $\log_a 49 = 2$ and ⓑ $\ln x = 3$.

Solution:

ⓐ

$$\log_a 49 = 2$$

Rewrite in exponential form. $a^2 = 49$

Solve the equation using the square root property. $a = \pm 7$

The base cannot be negative, so we eliminate $a = -7$. $a = 7, \quad \cancel{a = -7}$

Check.

$a = 7$

$$\begin{aligned} \log_a 49 &= 2 \\ \log_7 49 &\stackrel{?}{=} 2 \\ 7^2 &\stackrel{?}{=} 49 \\ 49 &= 49 \checkmark \end{aligned}$$

ⓑ

$$\ln x = 3$$

Rewrite in exponential form. $e^3 = x$

Check.

$x = e^3$

$$\begin{aligned} \ln x &= 3 \\ \ln e^3 &\stackrel{?}{=} 3 \\ e^3 &= e^3 \checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: ⓐ $\log_a 121 = 2$ ⓑ $\ln x = 7$

Solution:

ⓐ
$a = 11$
ⓑ $x = e^7$

Note:
Exercise:

Problem: Solve: ⓐ $\log_a 64 = 3$ ⓑ $\ln x = 9$

Solution:

ⓐ

$a = 4$
ⓑ $x = e^9$

Example:
Exercise:

Problem: Solve: ⓐ $\log_2(3x-5)=4$ and ⓑ $\ln e^{2x}=4$.

Solution:

ⓐ

	$\log_2(3x-5) = 4$
Rewrite in exponential form.	$2^4 = 3x-5$
Simplify.	$16 = 3x-5$
Solve the equation.	$21 = 3x$
	$7 = x$

Check.

$x = 7$

$$\begin{aligned} \log_2(3x-5) &= 4 \\ \log_2(3\cdot 7-5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16\checkmark \end{aligned}$$

ⓑ

	$\ln e^{2x} = 4$
Rewrite in exponential form.	$e^4 = e^{2x}$
Since the bases are the same the exponents are equal.	$4 = 2x$
Solve the equation.	$2 = x$

Check.

$x = 2$

$$\begin{aligned} \ln e^{2x} &= 4 \\ \ln e^{2\cdot 2} &\stackrel{?}{=} 4 \\ \ln e^4 &\stackrel{?}{=} 4 \\ e^4 &= e^4\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: ⓐ $\log_2(5x-1)=6$ ⓑ $\ln e^{3x}=6$

Solution:

ⓐ
$x = 13$
ⓑ $x = 2$

Note:
Exercise:

Problem: Solve: ⓐ $\log_3 (4x + 3) = 3$ ⓑ $\ln e^{4x} = 4$

Solution:

ⓐ
$x = 6$
ⓑ $x = 1$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Note:
Decibel Level of Sound
The loudness level, *D*, measured in decibels, of a sound of intensity, *I*, measured in watts per square inch is
Equation:

$$D = 10 \log \left(\frac{I}{10^{-12}} \right)$$

Example:
Exercise:

Problem:

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D = 10 \log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, *I*.	$D = 10 \log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10 \log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Note:
Exercise:

Problem:

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Solution:

The quiet dishwashers have a decibel level of 50 dB.

Note:
Exercise:

Problem: What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Solution:

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

Note:
Earthquake Intensity
The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Example:
Exercise:

Problem:

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.	$R = \log I$
1906 earthquake	$7.8 = \log I$
Convert to exponential form.	$I = 10^{7.8}$
2014 earthquake	$5.1 = \log I$
Convert to exponential form.	$I = 10^{5.1}$
Form a ratio of the intensities.	$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$
Substitute in the values.	$\frac{10^{7.8}}{10^{5.1}}$
Divide by subtracting the exponents.	$10^{2.7}$
Evaluate.	501
	The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

Note:
Exercise:

Problem:

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

Note:
Exercise:

Problem:

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the earthquake in Chile was about 1,259 times the intensity of the earthquake in Los Angeles.

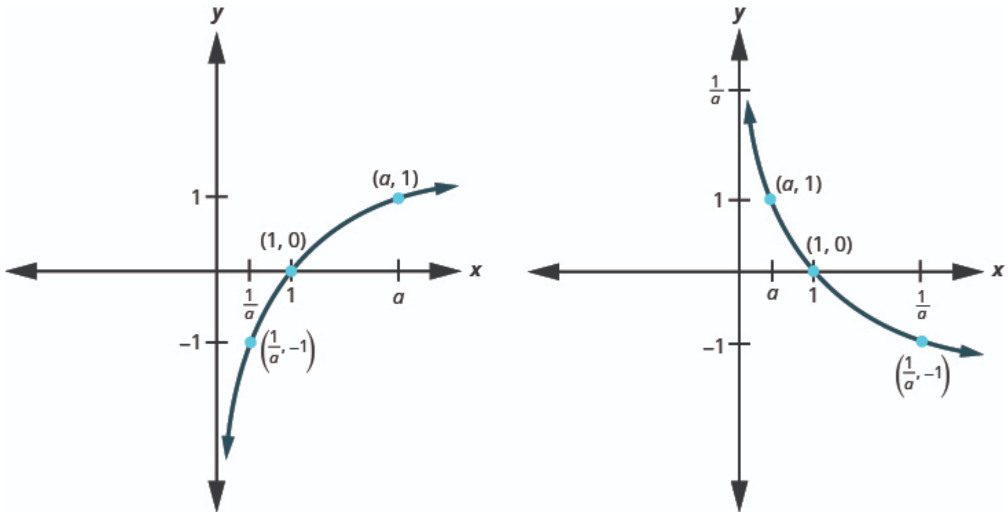
Note:
Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

Key Concepts

- **Properties of the Graph of** $y = \log_a x$:

when $a > 1$		**when** $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	*x*-intercept	$(1, 0)$
y-intercept	none	*y*-intercept	none
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	*y*-axis	Asymptote	*y*-axis
Basic shape	increasing	Basic shape	decreasing



- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10\log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Practice Makes Perfect

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

Exercise:

Problem: $4^2 = 16$

Exercise:

Problem: $2^5 = 32$

Solution:

$\log_2 32 = 5$

Exercise:

Problem: $3^3 = 27$

Exercise:

Problem: $5^3 = 125$

Solution:

$\log_5 125 = 3$

Exercise:

Problem: $10^3 = 1000$

Exercise:

Problem: $10^{-2} = \frac{1}{100}$

Solution:

$\log\frac{1}{100} = -2$

Exercise:

Problem: $x^{\frac{1}{2}} = \sqrt{3}$

Exercise:

Problem: $x^{\frac{1}{3}} = \sqrt[3]{6}$

Solution:

$\log_x\sqrt[3]{6} = \frac{1}{3}$

Exercise:

Problem: $32^x = \sqrt[4]{32}$

Exercise:

Problem: $17^x = \sqrt[5]{17}$

Solution:

$\log_{17}\sqrt[5]{17} = x$

Exercise:

Problem: $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Exercise:

Problem: $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

Solution:

$\log_{\frac{1}{3}}\frac{1}{81} = 4$

Exercise:

Problem: $3^{-2} = \frac{1}{9}$

Exercise:

Problem: $4^{-3} = \frac{1}{64}$

Solution:

$\log_4 \frac{1}{64} = -3$

Exercise:

Problem: $e^x = 6$

Exercise:

Problem: $e^3 = x$

Solution:

$\ln x = 3$

In the following exercises, convert each logarithmic equation to exponential form.

Exercise:

Problem: $3 = \log_4 64$

Exercise:

Problem: $6 = \log_2 64$

Solution:

$64 = 2^6$

Exercise:

Problem: $4 = \log_x 81$

Exercise:

Problem: $5 = \log_x 32$

Solution:

$32 = x^5$

Exercise:

Problem: $0 = \log_{12} 1$

Exercise:

Problem: $0 = \log_7 1$

Solution:

$1 = 7^0$

Exercise:

Problem: $1=\log_3 3$

Exercise:

Problem: $1=\log_9 9$

Solution:

$9=9^1$

Exercise:

Problem: $-4=\log_{10}\frac{1}{10,000}$

Exercise:

Problem: $3=\log_{10}1,000$

Solution:

$1,000=10^3$

Exercise:

Problem: $5=\log_e x$

Exercise:

Problem: $x=\log_e 43$

Solution:

$43=e^x$

Evaluate Logarithmic Functions

In the following exercises, find the value of x in each logarithmic equation.

Exercise:

Problem: $\log_x 49=2$

Exercise:

Problem: $\log_x 121=2$

Solution:

$x=11$

Exercise:

Problem: $\log_x 27=3$

Exercise:

Problem: $\log_x 64 = 3$

Solution:

$x = 4$

Exercise:

Problem: $\log_3 x = 4$

Exercise:

Problem: $\log_5 x = 3$

Solution:

$x = 125$

Exercise:

Problem: $\log_2 x = -6$

Exercise:

Problem: $\log_3 x = -5$

Solution:

$x = \frac{1}{243}$

Exercise:

Problem: $\log_{\frac{1}{4}} \frac{1}{16} = x$

Exercise:

Problem: $\log_{\frac{1}{3}} \frac{1}{9} = x$

Solution:

$x = 2$

Exercise:

Problem: $\log_{\frac{1}{4}} 64 = x$

Exercise:

Problem: $\log_{\frac{1}{9}} 81 = x$

Solution:

$x = -2$

In the following exercises, find the exact value of each logarithm without using a calculator.

Exercise:

Problem: $\log_7 49$

Exercise:

Problem: $\log_6 36$

Solution:

2

Exercise:

Problem: $\log_4 1$

Exercise:

Problem: $\log_5 1$

Solution:

0

Exercise:

Problem: $\log_{16} 4$

Exercise:

Problem: $\log_{27} 3$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\log_{\frac{1}{2}} 2$

Exercise:

Problem: $\log_{\frac{1}{2}} 4$

Solution:

-2

Exercise:

Problem: $\log_2 \frac{1}{16}$

Exercise:

Problem: $\log_3 \frac{1}{27}$

Solution:

-3

Exercise:

Problem: $\log_4 \frac{1}{16}$

Exercise:

Problem: $\log_9 \frac{1}{81}$

Solution:

-2

Graph Logarithmic Functions

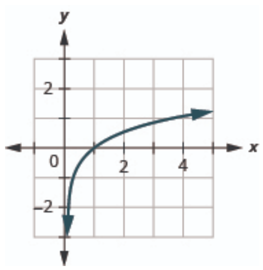
In the following exercises, graph each logarithmic function.
Exercise:

Problem: $y = \log_2 x$

Exercise:

Problem: $y = \log_4 x$

Solution:



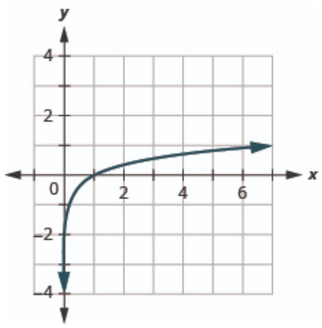
Exercise:

Problem: $y = \log_6 x$

Exercise:

Problem: $y = \log_7 x$

Solution:



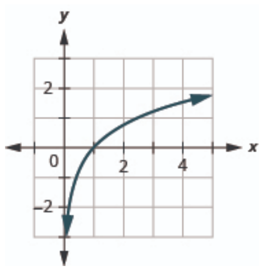
Exercise:

Problem: $y = \log_{1.5}x$

Exercise:

Problem: $y = \log_{2.5}x$

Solution:



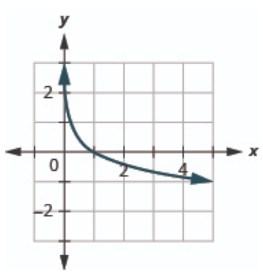
Exercise:

Problem: $y = \log_{\frac{1}{3}}x$

Exercise:

Problem: $y = \log_{\frac{1}{5}}x$

Solution:



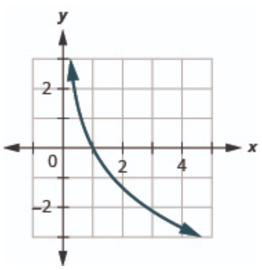
Exercise:

Problem: $y = \log_{0.4}x$

Exercise:

Problem: $y = \log_{0.6}x$

Solution:



Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

Exercise:

Problem: $\log_a 16 = 2$

Exercise:

Problem: $\log_a 81 = 2$

Solution:

$a = 9$

Exercise:

Problem: $\log_a 8 = 3$

Exercise:

Problem: $\log_a 27 = 3$

Solution:

$a = 3$

Exercise:

Problem: $\log_a 32 = 2$

Exercise:

Problem: $\log_a 24 = 3$

Solution:

$a = \sqrt[3]{24}$

Exercise:

Problem: $\ln x = 5$

Exercise:

Problem: $\ln x = 4$

Solution:

$x = e^4$

Exercise:

Problem: $\log_2 (5x + 1) = 4$

Exercise:

Problem: $\log_2 (6x + 2) = 5$

Solution:

$x = 5$

Exercise:

Problem: $\log_3 (4x - 3) = 2$

Exercise:

Problem: $\log_3 (5x - 4) = 4$

Solution:

$x = 17$

Exercise:

Problem: $\log_4 (5x + 6) = 3$

Exercise:

Problem: $\log_4 (3x - 2) = 2$

Solution:

$x = 6$

Exercise:

Problem: $\ln e^{4x} = 8$

Exercise:

Problem: $\ln e^{2x} = 6$

Solution:

$x = 3$

Exercise:

Problem: $\log x^2 = 2$

Exercise:

Problem: $\log(x^2 - 25) = 2$

Solution:

$x = -5\sqrt{5}, x = 5\sqrt{5}$

Exercise:

Problem: $\log_2\left(x^2 - 4\right) = 5$

Exercise:

Problem: $\log_3\left(x^2 + 2\right) = 3$

Solution:

$x = -5, x = 5$

Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

Exercise:

Problem: What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?

Exercise:

Problem: What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?

Solution:

A whisper has a decibel level of 20 dB.

Exercise:

Problem:

What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?

Exercise:

Problem:

What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?

Solution:

The sound of a garbage disposal has a decibel level of 100 dB.

Exercise:

Problem:

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.

Exercise:

Problem:

The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Solution:

The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

Writing Exercises

Exercise:

Problem: Explain how to change an equation from logarithmic form to exponential form.

Exercise:

Problem: Explain the difference between common logarithms and natural logarithms.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $\log_a a^x = x$.

Exercise:

Problem: Explain how to find the $\log_7 32$ on your calculator.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
convert between exponential and logarithmic form.			
evaluate logarithmic functions.			
graph logarithmic functions.			
solve logarithmic functions.			
use logarithmic models in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

common logarithmic function
: The function $f\left(x\right)=\log x$ is the common logarithmic function with base10, where $x>0$.
Equation:

$$y=\log x \text{ is equivalent to } x={10}^{y}$$

logarithmic function
: The function $f\left(x\right)={\log}_{a}x$ is the logarithmic function with base a, where $a>0$,$x>0$, and $a\ne 1$.
Equation:

$$y={\log}_{a}x \text{ is equivalent to } x={a}^{y}$$

natural logarithmic function
: The function $f\left(x\right)=\ln x$ is the natural logarithmic function with base e, where $x>0$.
Equation:

$$y=\ln x \text{ is equivalent to } x={e}^{y}$$

Use the Properties of Logarithms
By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Note:
Before you get started, take this readiness quiz.

1. Evaluate: ⓐ a^0 ⓑ a^1.
   If you missed this problem, review [link].
2. Write with a rational exponent: $\sqrt[3]{x^2 y}$.
   If you missed this problem, review [link].
3. Round to three decimal places: 2.5646415.
   If you missed this problem, review [link].

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1$, we can convert this to logarithmic form and get $\log_a 1 = 0$. Also, since $a^1 = a$, we get $\log_a a = 1$.

Note:
Properties of Logarithms
Equation:

$$\log_a 1 = 0 \qquad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $\log_8 1$ and ⓑ $\log_6 6$.

Solution:

ⓐ

	$\log_8 1$	
Use the property, $\log_a 1 = 0$.	0	$\log_8 1 = 0$

ⓑ

	$\log_6 6$	
Use the property, $\log_a a = 1$.	1	$\log_6 6 = 1$

Note:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $\log_{13} 1$ ⓑ $\log_9 9$.

Solution:

ⓐ 0 ⓑ 1

Note:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $\log_5 1$ ⓑ $\log_7 7$.

Solution:

ⓐ 0 ⓑ 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for *x* only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Note:
Inverse Properties of Logarithms
For $a > 0, x > 0$ and $a \neq 1$,
Equation:

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

Example:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $4^{\log_4 9}$ and ⓑ $\log_3 3^5$.

Solution:

ⓐ

	$4^{\log_4 9}$	
Use the property, $a^{\log_a x} = x$.	9	$4^{\log_4 9} = 9$

ⓑ

	$\log_3 3^5$	
Use the property, $a^{\log_a x} = x$.	5	$\log_3 3^5 = 5$

Note:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $5^{\log_5 15}$ ⓑ $\log_7 7^4$.

Solution:

ⓐ 15 ⓑ 4

Note:
Exercise:

Problem: Evaluate using the properties of logarithms: ⓐ $2^{\log_2 8}$ ⓑ $\log_2 2^{15}$.

Solution:

ⓐ 8 ⓑ 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Note:
Product Property of Logarithms
If $M > 0, N > 0$, a > 0 and a $\neq 1$, then,
Equation:

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example:
Exercise:

Problem:

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible: ⓐ $\log_3 7x$ and ⓑ $\log_4 64xy$.

Solution:

ⓐ

	$\log_3 7x$
Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.	$\log_3 7 + \log_3 x$
	$\log_3 7x = \log_3 7 + \log_3 x$

ⓑ

	$\log_4 64xy$
Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.	$\log_4 64 + \log_4 x + \log_4 y$
Simplify by evaluating $\log_4 64$.	$3 + \log_4 x + \log_4 y$
	$\log_4 64xy = 3 + \log_4 x + \log_4 y$

Note:
Exercise:

Problem:

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_3 3x$ ⓑ $\log_2 8xy$

Solution:

ⓐ $1 + \log_3 x$
ⓑ $3 + \log_2 x + \log_2 y$

Note:
Exercise:

Problem:

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

ⓐ $\log_9 9x$ ⓑ $\log_3 27xy$

Solution:

ⓐ $1+\log_9 x$
ⓑ $3+\log_3 x+\log_3 y$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n}=a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N}=\log_a M-\log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Note:
Quotient Property of Logarithms
If $M>0, N>0$,a >0 and a $\neq 1$, then,
Equation:

$$\log_a \frac{M}{N}=\log_a M-\log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M-\log_a N\neq \log_a(M-N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example:
Exercise:

Problem:

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.
ⓐ $\log_5 \frac{5}{7}$ and ⓑ $\log \frac{x}{100}$

Solution:

ⓐ

	$\log_5 \frac{5}{7}$
Use the Quotient Property, $\log_a \frac{M}{N}=\log_a M-\log_a N$.	$\log_5 5-\log_5 7$
Simplify.	$1-\log_5 7$
	$\log_5 \frac{5}{7}=1-\log_5 7$

ⓑ

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.
Simplify.

$\log \frac{x}{100}$
$\log x - \log 100$
$\log x - 2$

$\log \frac{x}{100} = \log x - 2$

Note:
Exercise:

Problem:

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_4 \frac{3}{4}$ ⓑ $\log \frac{x}{1000}$

Solution:

ⓐ $\log_4 3 - 1$ ⓑ $\log x - 3$

Note:
Exercise:

Problem:

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

ⓐ $\log_2 \frac{5}{4}$ ⓑ $\log \frac{10}{y}$

Solution:

ⓐ $\log_2 5 - 2$ ⓑ $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Note:
Power Property of Logarithms
If $M > 0$, $\text{a} > 0$, $\text{a} \neq 1$ and p is any real number then,
Equation:

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example:
Exercise:

Problem:

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.
ⓐ $\log_5 4^3$ and ⓑ $\log x^{10}$

Solution:

ⓐ

	$\log_5 4^3$
Use the Power Property, $\log_a M^p = p \log_a M$.	$3\log_5 4$
	$\log_5 4^3 = 3\log_5 4$

ⓑ

	$\log x^{10}$
Use the Power Property, $\log_a M^p = p \log_a M$.	$10 \log x$
	$\log x^{10} = 10\log x$

Note:
Exercise:

Problem:

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_7 5^4$ ⓑ $\log x^{100}$

Solution:

ⓐ $4\log_7 5$ ⓑ $100 \cdot \log x$

Note:
Exercise:

Problem:

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

ⓐ $\log_2 3^7$ ⓑ $\log x^{20}$

Solution:

ⓐ $7\log_2 3$ ⓑ $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Note:
Properties of Logarithms
If $M > 0,\ \mathrm{a} > 0,\ \mathrm{a} \neq 1$ and p is any real number then,

Property	**Base** a	**Base** e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln (M \ \cdot \ N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to “expand” a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_4 \left(2x^3y^2\right)$. Simplify, if possible.

Solution:

	$\log_4\left(2x^3y^2\right)$
Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$.	$\log_4 2 + \log_4 x^3 + \log_4 y^2$
Use the Power Property, $\log_a M^p = p \log_a M$, on the last two terms.	$\log_4 2 + 3\log_4 x + 2\log_4 y$
Simplify.	$\frac{1}{2} + 3\log_4 x + 2\log_4 y$
	$\log_4\left(2x^3y^2\right) = \frac{1}{2} + 3\log$

Note:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

Solution:

$\log_2 5 + 4\log_2 x + 2\log_2 y$

Note:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

Solution:

$\log_3 7 + 5\log_3 x + 3\log_3 y$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution:

	$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$
Rewrite the radical with a rational exponent.	$\log_2 \left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$
Use the Power Property, $\log_a M^p = p \log_a M$.	$\frac{1}{4}\log_2 \left(\frac{x^3}{3y^2z}\right)$
Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.	$\frac{1}{4}\left(\log_2 \left(x^3\right) - \log_2 \left(3y^2z\right)\right)$
Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N,$ in the second term.	$\frac{1}{4}\left(\log_2 \left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \text{lo}\right.\right.$
Use the Power Property, $\log_a M^p = p \log_a M,$ inside the parentheses.	$\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2$
Simplify by distributing.	$\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z$
	$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2$

Note:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

Solution:

$\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$

Note:
Exercise:

Problem: Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Solution:

$\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

Example:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution:

The log expressions all have the same base, 4.
The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.
Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

$\log_4 3 + \log_4 x - \log$

$\log_4 3x - \log_4 y$

$\log_4 \frac{3x}{y}$

$\log_4 3 + \log_4 x - \log$

Note:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Solution:

$\log_2 \frac{5x}{y}$

Note:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Solution:

$\log_3 \frac{6}{xy}$

Example:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3 (x+1)$. Simplify, if possible.

Solution:

The log expressions have the same base, 3.	$2\log_3 x + 4\log_3(x+1)$
Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$\log_3 x^2 + \log_3 (x+1)^4$
The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$\log_3 x^2 (x+1)^4$
	$2\log_3 x + 4\log_3(x+1) = \log_3 x^2 (x+1$

Note:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

Solution:

$\log_2 x^3 (x-1)^2$

Note:
Exercise:

Problem:

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Solution:

$\log x^2 (x+1)^2$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

Suppose we want to evaluate $\log_a M$.	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the expression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y \log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substitute $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Note:
Change-of-Base Formula
For any logarithmic bases a, b and $M > 0$,
Equation:

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b — new base 10 — new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

Example:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_4 35$.

Solution:

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify *a* and *M*. Choose 10 for *b*.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

Note:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_3 42$.

Solution:

3.402

Note:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_5 46$.

Solution:

2.379

Note:
Access these online resources for additional instruction and practice with using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

Key Concepts

- **Properties of Logarithms**
  Equation:

$$\log_a 1 = 0 \qquad \log_a a = 1$$

- **Inverse Properties of Logarithms**
  - For $a > 0$,$x > 0$ and $a \neq 1$
    Equation:

$$a^{\log_a x} = x \qquad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0$, $N > 0$, a > 0 and a $\neq 1$, then,
    Equation:

$$\log_a M \cdot N = \log_a M + \log_a N$$

    The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**
  - If $M > 0$, $N > 0$, a > 0 and a $\neq 1$, then,
    Equation:

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M > 0,\ \mathrm{a} > 0,\ \mathrm{a} \neq 1$ and p is any real number then,
    Equation:

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M > 0,\ \mathrm{a} > 0,\ \mathrm{a} \neq 1$ and p is any real number then,

Property	**Base** a	**Base** e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln (M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases *a* and *b*, and $M > 0$,
  Equation:

$$\begin{array}{ccc} \log_a M = \frac{\log_b M}{\log_b a} & \log_a M = \frac{\log M}{\log a} & \log_a M = \frac{\ln M}{\ln a} \\ \text{new base } b & \text{new base } 10 & \text{new base } e \end{array}$$

Practice Makes Perfect

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.
Exercise:

Problem: ⓐ $\log_4 1$ ⓑ $\log_8 8$

Exercise:

Problem: ⓐ $\log_{12} 1$ ⓑ $\ln e$

Solution:

ⓐ 0 ⓑ 1

Exercise:

Problem: ⓐ $3^{\log_3 6}$ ⓑ $\log_2 2^7$

Exercise:

Problem: ⓐ $5^{\log_5 10}$ ⓑ $\log_4 4^{10}$

Solution:

ⓐ 10 ⓑ 10

Exercise:

Problem: ⓐ $8^{\log_8 7}$ ⓑ $\log_6 6^{-2}$

Exercise:

Problem: ⓐ $6^{\log_6 15}$ ⓑ $\log_8 8^{-4}$

Solution:

ⓐ 15 ⓑ -4

Exercise:

Problem: ⓐ $10^{\log \sqrt{5}}$ ⓑ $\log 10^{-2}$

Exercise:

Problem: ⓐ $10^{\log \sqrt{3}}$ ⓑ $\log 10^{-1}$

Solution:

ⓐ $\sqrt{3}$ ⓑ -1

Exercise:

Problem: ⓐ $e^{\ln 4}$ ⓑ $\ln e^2$

Exercise:

Problem: ⓐ $e^{\ln 3}$ ⓑ $\ln e^7$

Solution:

ⓐ 3 ⓑ 7

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

Exercise:

Problem: $\log_4 6x$

Exercise:

Problem: $\log_5 8y$

Solution:

$\log_5 8 + \log_5 y$

Exercise:

Problem: $\log_2 32xy$

Exercise:

Problem: $\log_3 81xy$

Solution:

$4 + \log_3 x + \log_3 y$

Exercise:

Problem: $\log 100x$

Exercise:

Problem: $\log 1000y$

Solution:

$3 + \log y$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

Exercise:

Problem: $\log_3 \frac{3}{8}$

Exercise:

Problem: $\log_6 \frac{5}{6}$

Solution:

$\log_6 5 - 1$

Exercise:

Problem: $\log_4 \frac{16}{y}$

Exercise:

Problem: $\log_5 \frac{125}{x}$

Solution:

$3 - \log_5 x$

Exercise:

Problem: $\log \frac{x}{10}$

Exercise:

Problem: $\log \frac{10{,}000}{y}$

Solution:

$4 - \log y$

Exercise:

Problem: $\ln \frac{e^3}{3}$

Exercise:

Problem: $\ln \frac{e^4}{16}$

Solution:

$4 - \ln 16$

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

Exercise:

Problem: $\log_3 x^2$

Exercise:

Problem: $\log_2 x^5$

Solution:

$5\log_2 x$

Exercise:

Problem: $\log x^{-2}$

Exercise:

Problem: $\log x^{-3}$

Solution:

$-3\log x$

Exercise:

Problem: $\log_4\sqrt{x}$

Exercise:

Problem: $\log_5\sqrt[3]{x}$

Solution:

$\frac{1}{3}\log_5 x$

Exercise:

Problem: $\ln x^{\sqrt{3}}$

Exercise:

Problem: $\ln x^{\sqrt[3]{4}}$

Solution:

$\sqrt[3]{4}\ln x$

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

Exercise:

Problem: $\log_5\left(4x^6y^4\right)$

Exercise:

Problem: $\log_2\left(3x^5y^3\right)$

Solution:

$\log_2 3 + 5\log_2 x + 3\log_2 y$

Exercise:

Problem: $\log_3\left(\sqrt{2}x^2\right)$

Exercise:

Problem: $\log_5\left(\sqrt[4]{21}y^3\right)$

Solution:

$\frac{1}{4}\log_5 21 + 3\log_5 y$

Exercise:

Problem: $\log_3 \frac{xy^2}{z^2}$

Exercise:

Problem: $\log_5 \frac{4ab^3c^4}{d^2}$

Solution:

$\log_5 4 + \log_5 a + 3\log_5 b$
$+ 4\log_5 c - 2\log_5 d$

Exercise:

Problem: $\log_4 \frac{\sqrt{x}}{16y^4}$

Exercise:

Problem: $\log_3 \frac{\sqrt[3]{x^2}}{27y^4}$

Solution:

$\frac{2}{3}\log_3 x - 3 - 4\log_3 y$

Exercise:

Problem: $\log_2 \frac{\sqrt{2x+y^2}}{z^2}$

Exercise:

Problem: $\log_3 \frac{\sqrt{3x+2y^2}}{5z^2}$

Solution:

$\frac{1}{2}\log_3 \left(3x + 2y^2\right) - \log_3 5 - 2\log_3 z$

Exercise:

Problem: $\log_2 \sqrt[4]{\frac{5x^3}{2y^2z^4}}$

Exercise:

Problem: $\log_5 \sqrt[3]{\frac{3x^2}{4y^3z}}$

Solution:

$\frac{1}{3}(\log_5 3 + 2\log_5 x - \log_5 4$
$- 3\log_5 y - \log_5 z)$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.
Exercise:

Problem: $\log_6 4 + \log_6 9$

Exercise:

Problem: $\log 4 + \log 25$

Solution:

2

Exercise:

Problem: $\log_2 80 - \log_2 5$

Exercise:

Problem: $\log_3 36 - \log_3 4$

Solution:

2

Exercise:

Problem: $\log_3 4 + \log_3 (x+1)$

Exercise:

Problem: $\log_2 5 - \log_2 (x-1)$

Solution:

$\log_2 \frac{5}{x-1}$

Exercise:

Problem: $\log_7 3 + \log_7 x - \log_7 y$

Exercise:

Problem: $\log_5 2 - \log_5 x - \log_5 y$

Solution:

$\log_5 \frac{2}{xy}$

Exercise:

Problem: $4\log_2 x + 6\log_2 y$

Exercise:

Problem: $6\log_3 x + 9\log_3 y$

Solution:

$\log_3 x^6 y^9$

Exercise:

Problem: $\log_3\left(x^2-1\right)-2\log_3(x-1)$

Exercise:

Problem: $\log\left(x^2+2x+1\right)-2\log(x+1)$

Solution:

0

Exercise:

Problem: $4\log x-2\log y-3\log z$

Exercise:

Problem: $3\ln x+4\ln y-2\ln z$

Solution:

$\ln\frac{x^3y^4}{z^2}$

Exercise:

Problem: $\frac{1}{3}\log x-3\log(x+1)$

Exercise:

Problem: $2\log(2x+3)+\frac{1}{2}\log(x+1)$

Solution:

$\log(2x+3)^2\cdot\sqrt{x+1}$

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

Exercise:

Problem: $\log_3 42$

Exercise:

Problem: $\log_5 46$

Solution:

2.379

Exercise:

Problem: $\log_{12}87$

Exercise:

Problem: $\log_{15}93$

Solution:

1.674

Exercise:

Problem: $\log_{\sqrt{2}}17$

Exercise:

Problem: $\log_{\sqrt{3}}21$

Solution:

5.542

Writing Exercises

Exercise:

Problem:

Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x$,$\log_a (5+x)$. Why or why not?

Exercise:

Problem:

Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p$,$(\log_a x)^r$. Why or why not?

Solution:

Answers will vary.

Exercise:

Problem: Use an example to show that $\log (a+b) \neq \log a + \log b$?

Exercise:

Problem: Explain how to find the value of $\log_7 15$ using your calculator.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the properties of logarithms.			
use the Change of Base Formula.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Solve Exponential and Logarithmic Equations
By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

Note:
Before you get started, take this readiness quiz.

1. Solve: $x^2 = 16$.
   If you missed this problem, review [link].
2. Solve: $x^2 - 5x + 6 = 0$.
   If you missed this problem, review [link].
3. Solve: $x\left(x+6\right) = 2x + 5$.
   If you missed this problem, review [link].

Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N$. This is the **One-to-One Property of Logarithmic Equations**.

Note:
One-to-One Property of Logarithmic Equations
For $M > 0, N > 0,$ a>0, and a $\neq 1$ is any real number:
Equation:

$$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

Example:
Exercise:

Problem: Solve: $2\log_5 x = \log_5 81$.

Solution:

$$2\log_5 x = \log_5 81$$

Use the Power Property. $\log_5 x^2 = \log_5 81$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$. $x^2 = 81$

Solve using the Square Root Property. $x = \pm 9$

We eliminate $x = -9$ as we cannot take the logarithm of a negative number. $x = 9, \ \cancel{x = -9}$

Check.

$x = 9$

$$\begin{aligned} 2\log_5 x &= \log_5 81 \\ 2\log_5 9 &\stackrel{?}{=} \log_5 81 \\ \log_5 9^2 &\stackrel{?}{=} \log_5 81 \\ \log_5 81 &= \log_5 81 \checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: $2\log_3 x = \log_3 36$

Solution:

$x = 6$

Note:
Exercise:

Problem: Solve: $3\log x = \log 64$

Solution:

$x = 4$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

Example:
Exercise:

Problem: Solve: $\log_3 x + \log_3 (x - 8) = 2$.

Solution:

$$\log_3 x + \log_3(x-8) = 2$$

Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$. $\quad \log_3 x(x-8) = 2$

Rewrite in exponential form. $\quad 3^2 = x(x-8)$

Simplify. $\quad 9 = x^2 - 8x$

Subtract 9 from each side. $\quad 0 = x^2 - 8x - 9$

Factor. $\quad 0 = (x-9)(x +$

Use the Zero-Product Property. $\quad x - 9 = 0, \quad x + 1 =$

Solve each equation. $\quad x = 9, \quad \cancel{x = -}$

Check.

$x = -1$

$$\begin{aligned} \log_3 x + \log_3(x-8) &= 2 \\ \log_3(-1) + \log_3(-1-8) &\stackrel{?}{=} 2 \end{aligned}$$

We cannot take the log of a negative number.

$x = 9$

$$\begin{aligned} \log_3 x + \log_3(x-8) &= 2 \\ \log_3 9 + \log_3(9-8) &\stackrel{?}{=} 2 \\ 2 + 0 &\stackrel{?}{=} 2 \\ 2 &= 2 \checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: $\log_2 x + \log_2(x-2) = 3$

Solution:

$x = 4$

Note:
Exercise:

Problem: Solve: $\log_2 x + \log_2(x-6) = 4$

Solution:

$x = 8$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

Example:

Exercise:

Problem: Solve: $\log_4(x+6) - \log_4(2x+5) = -\log_4 x$.

Solution:

$$\log_4(x+6) - \log_4(2x+5) = -\mathrm{l}$$

Use the Quotient Property on the left side and the Power Property on the right.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \mathrm{log}$$

Rewrite $x^{-1} = \frac{1}{x}$.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \mathrm{log}$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$\frac{x+6}{2x+5} = \frac{1}{x}$$

Solve the rational equation.

$$x(x+6) = 2x$$

Distribute.

$$x^2 + 6x = 2x$$

Write in standard form.

$$x^2 + 4x - 5 = 0$$

Factor.

$$(x+5)(x-1) = 0$$

Use the Zero-Product Property.

$$x+5=0, \qquad x-1=0$$

Solve each equation.

$$\cancel{x=-5}, \qquad x=1$$

Check.

We leave the check for you.

Note:
Exercise:

Problem: Solve: $\log(x+2) - \log(4x+3) = -\log x$.

Solution:

$x = 3$

Note:
Exercise:

Problem: Solve: $\log(x-2) - \log(4x+16) = \log\frac{1}{x}$.

Solution:

$x = 8$

Solve Exponential Equations Using Logarithms

In the section on exponential functions, we solved some equations by writing both sides of the equation with the same base. Next we wrote a new equation by setting the exponents equal.

It is not always possible or convenient to write the expressions with the same base. In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

Example:
Exercise:

Problem: Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

Solution:

	5^x	$=$	11
Since the exponential is isolated, take the logarithm of both sides.	$\log 5^x$	$=$	$\log 11$
Use the Power Property to get the x as a factor, not an exponent.	$x \log 5$	$=$	$\log 11$
Solve for x. Find the exact answer.	x	$=$	$\frac{\log 11}{\log 5}$
Approximate the answer.	x	$\approx$	1.490

Since $5^1 = 5$ and $5^2 = 25,$ does it makes sense that $5^{1.490} \approx 11$?

Note:
Exercise:

Problem: Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

Solution:

$x = \frac{\log 43}{\log 7} \approx 1.933$

Note:
Exercise:

Problem: Solve $8^x = 98$. Find the exact answer and then approximate it to three decimal places.

Solution:

$x = \frac{\log 98}{\log 8} \approx 2.205$

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base *e*, we use the natural logarithm.

Example:
Exercise:

Problem: Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$\begin{array}{lrcl} & 3e^{x+2} & = & 24 \\ \text{Isolate the exponential by dividing both sides by 3.} & e^{x+2} & = & 8 \\ \text{Take the natural logarithm of both sides.} & \ln e^{x+2} & = & \ln 8 \\ \text{Use the Power Property to get the } x \text{ as a factor, not an exponent.} & (x+2)\ln e & = & \ln 8 \\ \text{Use the property } \ln e = 1 \text{ to simplify.} & x+2 & = & \ln 8 \\ \text{Solve the equation. Find the exact answer.} & x & = & \ln 8 - 2 \\ \text{Approximate the answer.} & x & \approx & 0.079 \end{array}$$

Note:
Exercise:

Problem: Solve $2e^{x-2} = 18$. Find the exact answer and then approximate it to three decimal places.

Solution:

$x = \ln 9 + 2 \approx 4.197$

Note:
Exercise:

Problem: Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Solution:

$x = \frac{\ln 5}{2} \approx 0.805$

Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

Note:
Compound Interest
For a principal, P, invested at an interest rate, r, for t years, the new balance, A is:
Equation:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	when compounded n times a year.
$A = Pe^{rt}$	when compounded continuously.

Example:
Exercise:

Problem:

Jermael's parents put $10,000 in investments for his college expenses on his first birthday. They hope the investments will be worth $50,000 when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution:

Identify the variables in the formula.	$\begin{aligned} A &= \$50{,}000 \\ P &= \$10{,}000 \\ r &= ? \\ t &= 17 \text{ years} \\ A &= Pe^{rt} \end{aligned}$
Substitute the values into the formula.	$50{,}000 = 10{,}000e^{r \cdot 17}$
Solve for r. Divide each side by 10,000.	$5 = e^{17r}$
Take the natural log of each side.	$\ln 5 = \ln e^{17r}$
Use the Power Property.	$\ln 5 = 17r \ln e$
Simplify.	$\ln 5 = 17r$
Divide each side by 17.	$\frac{\ln 5}{17} = r$
Approximate the answer.	$r \approx 0.095$
Convert to a percentage.	$r \approx 9.5\%$

They need the rate of growth to be approximately 9.5%

Note:
Exercise:

Problem:

Hector invests $10,000 at age 21. He hopes the investments will be worth $150,000 when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

Solution:

$r \approx 9.3\%$

Note:
Exercise:

Problem:

Rachel invests $\$15,000$ at age 25. She hopes the investments will be worth $\$90,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

Solution:

$r \approx 11.9\%$

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, k, and exponential decay has a negative rate of growth or decay constant, k.

Note:
Exponential Growth and Decay
For an original amount, A_0, that grows or decays at a rate, k, for a certain time, t, the final amount, A, is:
Equation:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

Example:
Exercise:

Problem:

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution:

This problem requires two main steps. First we must find the unknown rate, k. Then we use that value of k to help us find the unknown number of bacteria.

Identify the variables in the formula.	$\begin{array}{rcl} A & = & 300 \\ A_0 & = & 100 \\ k & = & ? \\ t & = & 3 \text{ hours} \end{array}$
	$A = A_0 e^{kt}$
Substitute the values in the formula.	$300 = 100e^{k \cdot 3}$
Solve for k. Divide each side by 100.	$3 = e^{3k}$
Take the natural log of each side.	$\ln 3 = \ln e^{3k}$
Use the Power Property.	$\ln 3 = 3k \ln e$
Simplify.	$\ln 3 = 3k$
Divide each side by 3.	$\frac{\ln 3}{3} = k$
Approximate the answer.	$k \approx 0.366$
We use this rate of growth to predict the number of bacteria there will be in 24 hours.	$\begin{array}{rcl} A & = & ? \\ A_0 & = & 100 \\ k & = & \frac{\ln 3}{3} \\ t & = & 24 \text{ hours} \end{array}$
	$A = A_0 e^{kt}$
Substitute in the values.	$A = 100e^{\frac{\ln 3}{3} \cdot 24}$
Evaluate.	$A \approx 656{,}100$
	At this rate of growth, they can expect 6

Note:
Exercise:

Problem:

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution:

There will be 62,500 bacteria.

Note:
Exercise:

Problem:

Researchers recorded that a certain bacteria population declined from 700,000 to 400,000 in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Solution:

There will be 5,870,061 bacteria.

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the half-life of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

Example:
Exercise:

Problem:

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

Solution:

This problem requires two main steps. First we must find the decay constant *k*. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find *k*. Then we use that value of *k* to help us find the amount of sample that will be left in 500 years.

Identify the variables in the formula.	$\begin{aligned} A &= 50 \\ A_0 &= 100 \\ k &= ? \\ t &= 1590\text{years} \\ A &= A_0 e^{kt} \end{aligned}$
Substitute the values in the formula.	$50 = 100e^{k \cdot 1590}$
Solve for k. Divide each side by 100.	$0.5 = e^{1590k}$
Take the natural log of each side.	$\ln 0.5 = \ln e^{1590k}$
Use the Power Property.	$\ln 0.5 = 1590k \ln e$
Simplify.	$\ln 0.5 = 1590k$
Divide each side by 1590.	$\frac{\ln 0.5}{1590} = k$ exact answer
We use this rate of growth to predict the amount that will be left in 500 years.	$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 0.5}{1590} \\ t &= 500\text{years} \\ A &= A_0 e^{kt} \end{aligned}$
Substitute in the values.	$A = 100e^{\frac{\ln 0.5}{1590} \cdot 500}$
Evaluate.	$A \approx 80.4\text{ mg}$
	In 500 years there would be approximately 80.4 mg remaining.

Note:
Exercise:

Problem:

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

Solution:

There will be 6.43 mg left.

Note:
Exercise:

Problem:

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

Solution:

There will be 31.5 mg left.

Note:
Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Logarithm Equations
- Finding the rate or time in a word problem on exponential growth or decay
- Finding the rate or time in a word problem on exponential growth or decay

Key Concepts

- **One-to-One Property of Logarithmic Equations:** For $M > 0, N > 0,$ a>0, and a $\neq 1$ is any real number:
  Equation:

  $$\text{If } \log_a M = \log_a N, \text{ then } M = N.$$

- **Compound Interest:**
  For a principal, *P*, invested at an interest rate, *r*, for *t* years, the new balance, *A*, is:
  Equation:

  $$\begin{array}{ll} A = P\left(1 + \frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A = Pe^{rt} & \text{when compounded continuously.} \end{array}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, *r*, for a certain time *t*, the final amount, *A*, is $A = A_0 e^{rt}$.

Section Exercises

Practice Makes Perfect

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for *x*.

Exercise:

Problem: $\log_4 64 = 2\log_4 x$

Exercise:

Problem: $\log 49 = 2\log x$

Solution:

$x = 7$

Exercise:

Problem: $3\log_3 x = \log_3 27$

Exercise:

Problem: $3\log_6 x = \log_6 64$

Solution:

$x = 4$

Exercise:

Problem: $\log_5 (4x - 2) = \log_5 10$

Exercise:

Problem: $\log_3 \left(x^2 + 3\right) = \log_3 4x$

Solution:

$x = 1, x = 3$

Exercise:

Problem: $\log_3 x + \log_3 x = 2$

Exercise:

Problem: $\log_4 x + \log_4 x = 3$

Solution:

$x = 8$

Exercise:

Problem: $\log_2 x + \log_2 (x - 3) = 2$

Exercise:

Problem: $\log_3 x + \log_3 (x + 6) = 3$

Solution:

$x = 3$

Exercise:

Problem: $\log x + \log (x + 3) = 1$

Exercise:

Problem: $\log x + \log (x - 15) = 2$

Solution:

$x = 20$

Exercise:

Problem: $\log (x + 4) - \log (5x + 12) = -\log x$

Exercise:

Problem: $\log (x - 1) - \log (x + 3) = \log \frac{1}{x}$

Solution:

$x = 3$

Exercise:

Problem: $\log_5 (x + 3) + \log_5 (x - 6) = \log_5 10$

Exercise:

Problem: $\log_5 (x + 1) + \log_5 (x - 5) = \log_5 7$

Solution:

$x = 6$

Exercise:

Problem: $\log_3 (2x - 1) = \log_3 (x + 3) + \log_3 3$

Exercise:

Problem: $\log (5x + 1) = \log (x + 3) + \log 2$

Solution:

$x = \frac{5}{3}$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

Exercise:

Problem: $3^x = 89$

Exercise:

Problem: $2^x = 74$

Solution:

$x = \frac{\log 74}{\log 2} \approx 6.209$

Exercise:

Problem: $5^x = 110$

Exercise:

Problem: $4^x = 112$

Solution:

$x = \frac{\log 112}{\log 4} \approx 3.404$

Exercise:

Problem: $e^x = 16$

Exercise:

Problem: $e^x = 8$

Solution:

$x = \ln 8 \approx 2.079$

Exercise:

Problem: $\left(\frac{1}{2}\right)^x = 6$

Exercise:

Problem: $\left(\frac{1}{3}\right)^x = 8$

Solution:

$x = \frac{\log 8}{\log \frac{1}{3}} \approx -1.893$

Exercise:

Problem: $4e^{x+1} = 16$

Exercise:

Problem: $3e^{x+2} = 9$

Solution:

$x = \ln 3 - 2 \approx -0.901$

Exercise:

Problem: $6e^{2x}=24$

Exercise:

Problem: $2e^{3x}=32$

Solution:

$x=\frac{\ln 16}{3}\approx 0.924$

Exercise:

Problem: $\frac{1}{4}e^{x}=3$

Exercise:

Problem: $\frac{1}{3}e^{x}=2$

Solution:

$x=\ln 6\approx 1.792$

Exercise:

Problem: $e^{x+1}+2=16$

Exercise:

Problem: $e^{x-1}+4=12$

Solution:

$x=\ln 8+1\approx 3.079$

In the following exercises, solve each equation.

Exercise:

Problem: $3^{3x+1}=81$

Exercise:

Problem: $6^{4x-17}=216$

Solution:

$x=5$

Exercise:

Problem: $\frac{e^{x^2}}{e^{14}}=e^{5x}$

Exercise:

Problem: $\frac{e^{x^2}}{e^{x}}=e^{20}$

Solution:

$x = -4, x = 5$

Exercise:

Problem: $\log_a 64 = 2$

Exercise:

Problem: $\log_a 81 = 4$

Solution:

$a = 3$

Exercise:

Problem: $\ln x = -8$

Exercise:

Problem: $\ln x = 9$

Solution:

$x = e^9$

Exercise:

Problem: $\log_5 (3x - 8) = 2$

Exercise:

Problem: $\log_4 (7x + 15) = 3$

Solution:

$x = 7$

Exercise:

Problem: $\ln e^{5x} = 30$

Exercise:

Problem: $\ln e^{6x} = 18$

Solution:

$x = 3$

Exercise:

Problem: $3\log x = \log 125$

Exercise:

Problem: $7\log_3 x = \log_3 128$

Solution:

$x = 2$

Exercise:

Problem: $\log_6 x + \log_6 (x - 5) = 24$

Exercise:

Problem: $\log_9 x + \log_9 (x - 4) = 12$

Solution:

$x = 6$

Exercise:

Problem: $\log_2 (x + 2) - \log_2 (2x + 9) = -\log_2 x$

Exercise:

Problem: $\log_6 (x + 1) - \log_6 (4x + 10) = \log_6 \frac{1}{x}$

Solution:

$x = 5$

In the following exercises, solve for *x*, giving an exact answer as well as an approximation to three decimal places.

Exercise:

Problem: $6^x = 91$

Exercise:

Problem: $\left(\frac{1}{2}\right)^x = 10$

Solution:

$x = \frac{\log 10}{\log \frac{1}{2}} \approx -3.322$

Exercise:

Problem: $7e^{x-3} = 35$

Exercise:

Problem: $8e^{x+5} = 56$

Solution:

$x = \ln 7 - 5 \approx -3.054$

Use Exponential Models in Applications

In the following exercises, solve.

Exercise:

Problem:

Sung Lee invests $\$5,000$ at age 18. He hopes the investments will be worth $\$10,000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

Exercise:

Problem:

Alice invests $\$15,000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

Solution:

6.9%

Exercise:

Problem:

Coralee invests $\$5,000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?

Exercise:

Problem:

Simone invests $\$8,000$ in an account that compounds interest quarterly and earns 5%. How long will it take for his money to double?

Solution:

13.9 years

Exercise:

Problem:

Researchers recorded that a certain bacteria population declined from 100,000 to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?

Exercise:

Problem:

Researchers recorded that a certain bacteria population declined from 800,000 to 500,000 in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?

Solution:

122,070 bacteria

Exercise:

Problem:

A virus takes 6 days to double its original population $(A = 2A_0)$. How long will it take to triple its population?

Exercise:

Problem:

A bacteria doubles its original population in 24 hours $(A = 2A_0)$. How big will its population be in 72 hours?

Solution:

8 times as large as the original population

Exercise:

Problem:

Carbon-14 is used for archeological carbon dating. Its half-life is 5,730 years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?

Exercise:

Problem:

Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

Solution:

0.03 ml

Writing Exercises

Exercise:

Problem:

Explain the method you would use to solve these equations: $3^{x+1} = 81$, $3^{x+1} = 75$. Does your method require logarithms for both equations? Why or why not?

Exercise:

Problem:

What is the difference between the equation for exponential growth versus the equation for exponential decay?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve logarithmic equations using the properties of logarithms.			
solve exponential equations using logarithms.			
use exponential models in applications.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

Finding Composite and Inverse Functions

Find and Evaluate Composite Functions

In the following exercises, for each pair of functions, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Exercise:

Problem: $f(x) = 7x - 2$ and $g(x) = 5x + 1$

Exercise:

Problem: $f(x) = 4x$ and $g(x) = x^2 + 3x$

Solution:

ⓐ $4x^2 + 12x$ ⓑ $16x^2 + 12x$ ⓒ $4x^3 + 12x^2$

In the following exercises, evaluate the composition.

Exercise:

Problem: For functions $f(x) = 3x^2 + 2$ and $g(x) = 4x - 3$, find ⓐ $(f \circ g)(-3)$ ⓑ $(g \circ f)(-2)$ ⓒ $(f \circ f)(-1)$

Exercise:

Problem: For functions $f(x) = 2x^3 + 5$ and $g(x) = 3x^2 - 7$, find ⓐ $(f \circ g)(-1)$ ⓑ $(g \circ f)(-2)$ ⓒ $(g \circ g)(1)$

Solution:

ⓐ -123 ⓑ 356 ⓒ 41

Determine Whether a Function is One-to-One

In the following exercises, for each set of ordered pairs, determine if it represents a function and if so, is the function one-to-one.

Exercise:

Problem: $\{(-3,-5),(-2,-4),(-1,-3),(0,-2),$ $(-1,-1),(-2,0),(-3,1)\}$

Exercise:

Problem: $\{(-3,0),(-2,-2),(-1,0),(0,1),$ $(1,2),(2,1),(3,-1)\}$

Solution:

Function; not one-to-one

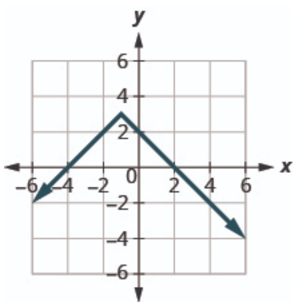
Exercise:

Problem: $\{(-3,3),(-2,1),(-1,-1),(0,-3),$ $(1,-5),(2,-4),(3,-2)\}$

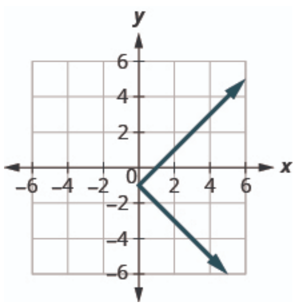
In the following exercises, determine whether each graph is the graph of a function and if so, is it one-to-one.

Exercise:

Problem: ⓐ



ⓑ

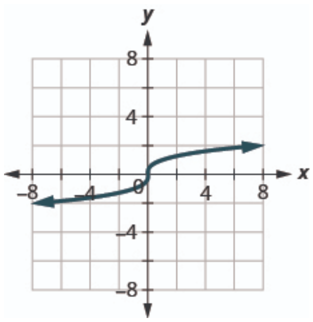


Solution:

ⓐ Function; not one-to-one ⓑ Not a function

Exercise:

Problem: ⓐ



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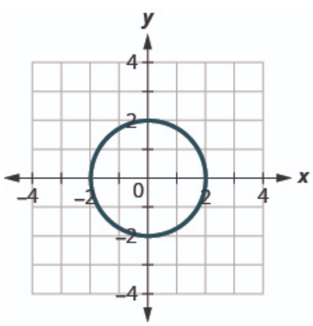


Find the Inverse of a Function

In the following exercise, find the inverse of the function. Determine the domain and range of the inverse function.
Exercise:

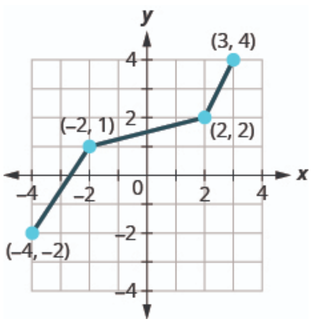
Problem: $\{(-3,10),(-2,5),(-1,2),(0,1)\}$

Solution:

Inverse function: $\{(10,-3),(5,-2),(2,-1),(1,0)\}$. Domain: $\{1,2,5,10\}$. Range: $\{-3,-2,-1,0\}$.

In the following exercise, graph the inverse of the one-to-one function shown.
Exercise:

Problem:



In the following exercises, verify that the functions are inverse functions.

Exercise:

Problem: $f\left(x\right)=3x+7$ and $g\left(x\right)=\frac{x-7}{3}$

Solution:

$g(f(x))=x$, and $f(g(x))=x$, so they are inverses.

Exercise:

Problem: $f\left(x\right)=2x+9$ and $g\left(x\right)=\frac{x+9}{2}$

In the following exercises, find the inverse of each function.

Exercise:

Problem: $f\left(x\right)=6x-11$

Solution:

$f^{-1}\left(x\right)=\frac{x+11}{6}$

Exercise:

Problem: $f\left(x\right)=x^{3}+13$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x+5}$

Solution:

$f^{-1}\left(x\right)=\frac{1}{x}-5$

Exercise:

Problem: $f\left(x\right)=\sqrt[5]{x-1}$

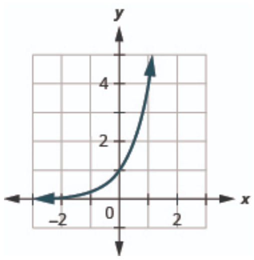
Evaluate and Graph Exponential Functions

Graph Exponential Functions

In the following exercises, graph each of the following functions.
Exercise:

Problem: $f\left(x\right)=4^x$

Solution:



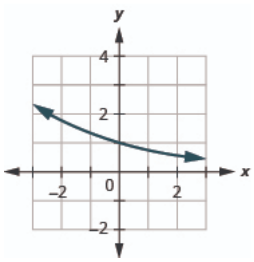
Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{5}\right)^x$

Exercise:

Problem: $g\left(x\right)=\left(0.75\right)^x$

Solution:



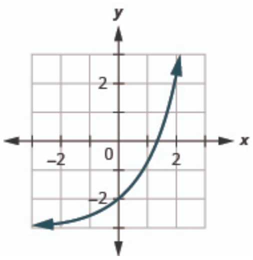
Exercise:

Problem: $g\left(x\right)=3^{x+2}$

Exercise:

Problem: $f\left(x\right)=\left(2.3\right)^x-3$

Solution:



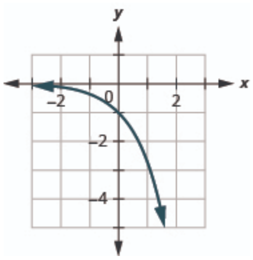
Exercise:

Problem: $f\left(x\right)=e^{x}+5$

Exercise:

Problem: $f\left(x\right)=-e^{x}$

Solution:



Solve Exponential Equations

In the following exercises, solve each equation.

Exercise:

Problem: $3^{5x-6}=81$

Exercise:

Problem: $2^{x^{2}}=16$

Solution:

$x=-2,x=2$

Exercise:

Problem: $9^{x}=27$

Exercise:

Problem: $5^{x^{2}+2x}=\frac{1}{5}$

Solution:

$x=-1$

Exercise:

Problem: $e^{4x} \cdot e^{7} = e^{19}$

Exercise:

Problem: $\frac{e^{x^2}}{e^{15}} = e^{2x}$

Solution:

$x = -3, x = 5$

Use Exponential Models in Applications

In the following exercises, solve.

Exercise:

Problem:

Felix invested $\$12{,}000$ in a savings account. If the interest rate is 4% how much will be in the account in 12 years by each method of compounding?

ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously.

Exercise:

Problem:

Sayed deposits $\$20{,}000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning 7% per year and is compounded continuously?

Solution:

$163,323.40

Exercise:

Problem:

A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of 15% per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?

Exercise:

Problem:

In the last five years the population of the United States has grown at a rate of 0.7% per year to about 318,900,000. If this rate continues, what will be the population in 5 more years?

Solution:

330,259,000

Evaluate and Graph Logarithmic Functions

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.
Exercise:

Problem: $5^4 = 625$

Exercise:

Problem: $10^{-3} = \frac{1}{1{,}000}$

Solution:

$\log \frac{1}{1{,}000} = -3$

Exercise:

Problem: $63^{\frac{1}{5}} = \sqrt[5]{63}$

Exercise:

Problem: $e^y = 16$

Solution:

$\ln 16 = y$

In the following exercises, convert each logarithmic equation to exponential form.
Exercise:

Problem: $7 = \log_2 128$

Exercise:

Problem: $5 = \log 100{,}000$

Solution:

$100000 = 10^5$

Exercise:

Problem: $4 = \ln x$

Evaluate Logarithmic Functions

In the following exercises, solve for *x*.
Exercise:

Problem: $\log_x 125 = 3$

Solution:

$x = 5$

Exercise:

Problem: $\log_7 x = -2$

Exercise:

Problem: $\log_{\frac{1}{2}} \frac{1}{16} = x$

Solution:

$x = 4$

In the following exercises, find the exact value of each logarithm without using a calculator.

Exercise:

Problem: $\log_2 32$

Exercise:

Problem: $\log_8 1$

Solution:

0

Exercise:

Problem: $\log_3 \frac{1}{9}$

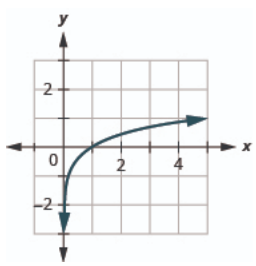
Graph Logarithmic Functions

In the following exercises, graph each logarithmic function.

Exercise:

Problem: $y = \log_5 x$

Solution:



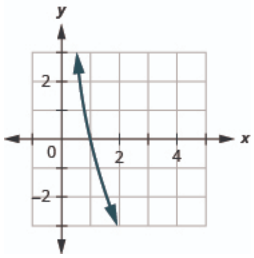
Exercise:

Problem: $y = \log_{\frac{1}{4}} x$

Exercise:

Problem: $y = \log_{0.8} x$

Solution:



Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

Exercise:

Problem: $\log_a 36 = 5$

Exercise:

Problem: $\ln x = -3$

Solution:

$x = e^{-3}$

Exercise:

Problem: $\log_2 (5x - 7) = 3$

Exercise:

Problem: $\ln e^{3x} = 24$

Solution:

$x = 8$

Exercise:

Problem: $\log(x^2 - 21) = 2$

Use Logarithmic Models in Applications

Exercise:

Problem: What is the decibel level of a train whistle with intensity 10^{-3} watts per square inch?

Solution:

90 dB

[Use the Properties of Logarithms]

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

Exercise:

Problem: ⓐ $\log_7 1$ ⓑ $\log_{12} 12$

Exercise:

Problem: ⓐ $5^{\log_5 13}$ ⓑ $\log_3 3^{-9}$

Solution:

ⓐ 13 ⓑ -9

Exercise:

Problem: ⓐ $10^{\log \sqrt{5}}$ ⓑ $\log 10^{-3}$

Exercise:

Problem: ⓐ $e^{\ln 8}$ ⓑ $\ln e^5$

Solution:

ⓐ 8 ⓑ 5

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

Exercise:

Problem: $\log_4 (64xy)$

Exercise:

Problem: $\log 10{,}000m$

Solution:

$4 + \log m$

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

Exercise:

Problem: $\log_7 \frac{49}{y}$

Exercise:

Problem: $\ln \frac{e^5}{2}$

Solution:

$5 - \ln 2$

In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.
Exercise:

Problem: $\log x^{-9}$

Exercise:

Problem: $\log_4 \sqrt[7]{z}$

Solution:

$\frac{1}{7}\log_4 z$

In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.
Exercise:

Problem: $\log_3 \left(\sqrt{4}x^7y^8\right)$

Exercise:

Problem: $\log_5 \frac{8a^2b^6c}{d^3}$

Solution:

$\log_5 8 + 2\log_5 a + 6\log_5 b$
$+ \log_5 c - 3\log_5 d$

Exercise:

Problem: $\ln \frac{\sqrt{3x^2-y^2}}{z^4}$

Exercise:

Problem: $\log_6 \sqrt[3]{\frac{7x^2}{6y^3z^5}}$

Solution:

$\frac{1}{3}(\log_6 7 + 2\log_6 x - 1 - 3\log_6 y$
$- 5\log_6 z)$

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.
Exercise:

Problem: $\log_2 56 - \log_2 7$

Exercise:

Problem: $3\log_3 x + 7\log_3 y$

Solution:

$\log_3 x^3y^7$

Exercise:

Problem: $\log_5 \left(x^2 - 16\right) - 2\log_5 (x + 4)$

Exercise:

Problem: $\frac{1}{4}\log y - 2\log (y - 3)$

Solution:

$\log\frac{\sqrt[4]{y}}{(y-3)^2}$

Use the Change-of-Base Formula

In the following exercises, rounding to three decimal places, approximate each logarithm.

Exercise:

Problem: $\log_5 97$

Exercise:

Problem: $\log_{\sqrt{3}} 16$

Solution:

5.047

[Solve Exponential and Logarithmic Equations]

Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for *x*.

Exercise:

Problem: $3\log_5 x = \log_5 216$

Exercise:

Problem: $\log_2 x + \log_2 (x - 2) = 3$

Solution:

$x = 4$

Exercise:

Problem: $\log (x - 1) - \log (3x + 5) = -\log x$

Exercise:

Problem: $\log_4 (x - 2) + \log_4 (x + 5) = \log_4 8$

Solution:

$x = 3$

Exercise:

Problem: $\ln(3x - 2) = \ln(x + 4) + \ln 2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

Exercise:

Problem: $2^x = 101$

Solution:

$x = \frac{\log 101}{\log 2} \approx 6.658$

Exercise:

Problem: $e^x = 23$

Exercise:

Problem: $\left(\frac{1}{3}\right)^x = 7$

Solution:

$x = \frac{\log 7}{\log \frac{1}{3}} \approx -1.771$

Exercise:

Problem: $7e^{x+3} = 28$

Exercise:

Problem: $e^{x-4} + 8 = 23$

Solution:

$x = \ln 15 + 4 \approx 6.708$

Use Exponential Models in Applications

Exercise:

Problem:

Jerome invests $\$18,000$ at age 17. He hopes the investments will be worth $\$30,000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?

Exercise:

Problem:

Elise invests $\$4500$ in an account that compounds interest monthly and earns 6%. How long will it take for her money to double?

Solution:

11.6 years

Exercise:

Problem:

Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?

Exercise:

Problem:

Mouse populations can double in 8 months $(A = 2A_0)$. How long will it take for a mouse population to triple?

Solution:

12.7 months

Exercise:

Problem: The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

Practice Test

Exercise:

Problem:

For the functions, $f(x) = 6x + 1$ and $g(x) = 8x - 3$, find ⓐ $(f \circ g)(x)$, ⓑ $(g \circ f)(x)$, and ⓒ $(f \cdot g)(x)$.

Solution:

ⓐ $48x - 17$ ⓑ $48x + 5$
ⓒ $48x^2 - 10x - 3$

Exercise:

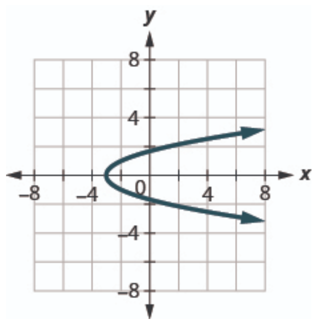
Problem:

Determine if the following set of ordered pairs represents a function and if so, is the function one-to-one.
$\{(-2, 2), (-1, -3), (0, 1), (1, -2), (2, -3)\}$

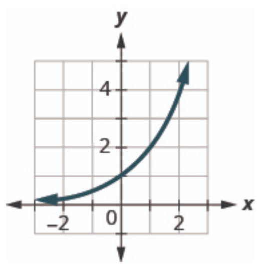
Exercise:

Problem: Determine whether each graph is the graph of a function and if so, is it one-to-one.

ⓐ



ⓑ

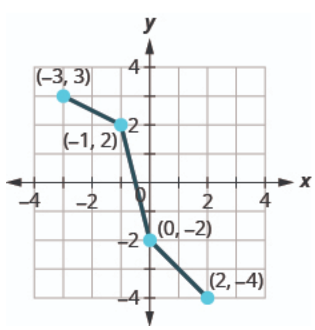


Solution:

ⓐ Not a function ⓑ One-to-one function

Exercise:

Problem: Graph, on the same coordinate system, the inverse of the one-to-one function shown.



Exercise:

Problem: Find the inverse of the function $f\left(x\right)={x}^{5}-9$.

Solution:

${f}^{-1}\left(x\right)=\sqrt[5]{x+9}$

Exercise:

Problem: Graph the function $g\left(x\right)={2}^{x-3}$.

Exercise:

Problem: Solve the equation $2^{2x-4} = 64$.

Solution:

$x = 5$

Exercise:

Problem: Solve the equation $\frac{e^{x^2}}{e^4} = e^{3x}$.

Exercise:

Problem:

Megan invested $\$21,000$ in a savings account. If the interest rate is 5%, how much will be in the account in 8 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously.

Solution:

ⓐ $\$31,250.74$ ⓑ $\$31,302.29$ ⓒ $\$31,328.32$

Exercise:

Problem: Convert the equation from exponential to logarithmic form: $10^{-2} = \frac{1}{100}$.

Exercise:

Problem: Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$

Solution:

$343 = 7^3$

Exercise:

Problem: Solve for *x*: $\log_5 x = -3$

Exercise:

Problem: Evaluate $\log_{11} 1$.

Solution:

0

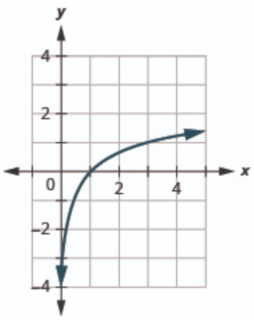
Exercise:

Problem: Evaluate $\log_4 \frac{1}{64}$.

Exercise:

Problem: Graph the function $y = \log_3 x$.

Solution:



Exercise:

Problem: Solve for *x*: $\log(x^2 - 39) = 1$

Exercise:

Problem: What is the decibel level of a small fan with intensity 10^{-8} watts per square inch?

Solution:

40 dB

Exercise:

Problem: Evaluate each. ⓐ $6^{\log_6 17}$ ⓑ $\log_9 9^{-3}$

In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.

Exercise:

Problem: $\log_5 25ab$

Solution:

$2 + \log_5 a + \log_5 b$

Exercise:

Problem: $\ln \frac{e^{12}}{8}$

Exercise:

Problem: $\log_2 \sqrt[4]{\frac{5x^3}{16y^2z^7}}$

Solution:

$\frac{1}{4}(\log_2 5 + 3\log_2 x - 4 - 2\log_2 y$
$- 7\log_2 z)$

In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.

Exercise:

Problem: $5\log_4 x + 3\log_4 y$

Exercise:

Problem: $\frac{1}{6}\log x - 3\log(x+5)$

Solution:

$\log\frac{\sqrt[6]{x}}{(x+5)^3}$

Exercise:

Problem: Rounding to three decimal places, approximate $\log_4 73$.

Exercise:

Solve for *x*:

Problem: $\log_7(x+2) + \log_7(x-3) = \log_7 24$

Solution:

$x = 6$

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

Exercise:

Problem: $\left(\frac{1}{5}\right)^x = 9$

Exercise:

Problem: $5e^{x-4} = 40$

Solution:

$x = \ln 8 + 4 \approx 6.079$

Exercise:

Problem:

Jacob invests $14,000 in an account that compounds interest quarterly and earns 4%. How long will it take for his money to double?

Exercise:

Problem:

Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?

Solution:

1,921 bacteria

Exercise:

Problem:

A certain beetle population can double in 3 months $(A = 2A_0)$. How long will it take for that beetle population to triple?

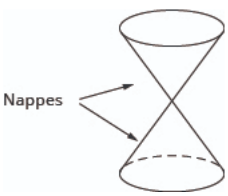
Distance and Midpoint Formulas; Circles
By the end of this section, you will be able to:

- Use the Distance Formula
- Use the Midpoint Formula
- Write the equation of a circle in standard form
- Graph a circle

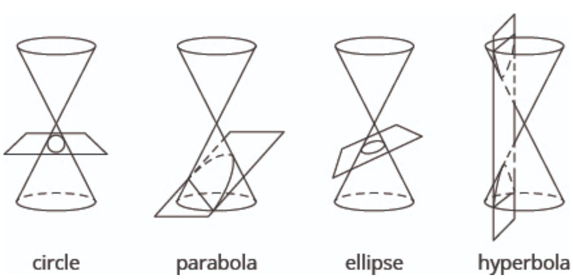
Note:
Before you get started, take this readiness quiz.

1. Find the length of the hypotenuse of a right triangle whose legs are 12 and 16 inches.
   If you missed this problem, review [link].
2. Factor: $x^2 - 18x + 81$.
   If you missed this problem, review [link].
3. Solve by completing the square: $x^2 - 12x - 12 = 0$.
   If you missed this problem, review [link].

In this chapter we will be looking at the conic sections, usually called the conics, and their properties. The conics are curves that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.



There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola**. The next figure shows how the plane intersecting the double cone results in each curve.



Each of the curves has many applications that affect your daily life, from your cell phone to acoustics and navigation systems. In this section we will look at the properties of a circle.

Use the Distance Formula

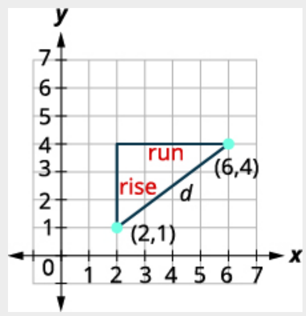
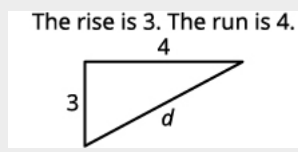
We have used the Pythagorean Theorem to find the lengths of the sides of a right triangle. Here we will use this theorem again to find distances on the rectangular coordinate system. By finding distance on the rectangular coordinate system, we can make a connection between the geometry of a conic and algebra—which opens up a world of opportunities for application.

Our first step is to develop a formula to find distances between points on the rectangular coordinate system. We will plot the points and create a right triangle much as we did when we found slope in Graphs and Functions. We then take it one step further and use the Pythagorean Theorem to find the length of the hypotenuse of the triangle —which is the distance between the points.

Example:
Exercise:

Problem: Use the rectangular coordinate system to find the distance between the points $(6, 4)$ and $(2, 1)$.

Solution:

Plot the two points. Connect the two points with a line. Draw a right triangle as if you were going to find slope.	
Find the length of each leg.	
Use the Pythagorean Theorem to find *d*, the distance between the two points.	$a^2 + b^2 = c^2$
Substitute in the values.	$3^2 + 4^2 = d^2$
Simplify.	$9 + 16 = d^2$
	$25 = d^2$
Use the Square Root Property.	$d = 5 \qquad \cancel{d = -5}$
Since distance, *d* is positive, we can eliminate $d = -5$.	The distance between the points $(6, 4)$ and $(2, 1)$ is 5.

Note:
Exercise:

Problem: Use the rectangular coordinate system to find the distance between the points $(6,1)$ and $(2,-2)$.

Solution:

$d=5$

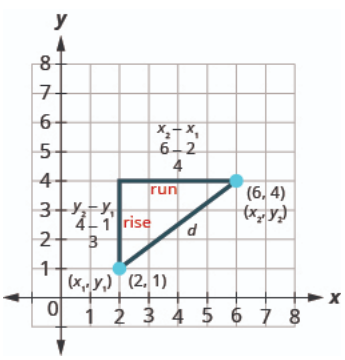
Note:
Exercise:

Problem:

Use the rectangular coordinate system to find the distance between the points $(5,3)$ and $(-3,-3)$.

Solution:

$d=10$



The method we used in the last example leads us to the formula to find the distance between the two points (x_1,y_1) and (x_2,y_2).

When we found the length of the horizontal leg we subtracted $6-2$ which is x_2-x_1.

When we found the length of the vertical leg we subtracted $4-1$ which is y_2-y_1.

If the triangle had been in a different position, we may have subtracted x_1-x_2 or y_1-y_2. The expressions x_2-x_1 and x_1-x_2 vary only in the sign of the resulting number. To get the positive value-since distance is positive- we can use absolute value. So to generalize we will say $|x_2-x_1|$ and $|y_2-y_1|$.

In the Pythagorean Theorem, we substitute the general expressions $|x_2-x_1|$ and $|y_2-y_1|$ rather than the numbers.

$$a^2 + b^2 = c^2$$

Substitute in the values. $$(|x_2 - x_1|)^2 + (|y_2 - y_1|)^2 = d^2$$

Squaring the expressions makes them positive, so we eliminate the absolute value bars. $$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

Use the Square Root Property. $$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distance is positive, so eliminate the negative value. $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is the Distance Formula we use to find the distance *d* between the two points (x_1, y_1) and (x_2, y_2).

Note:
Distance Formula
The distance *d* between the two points (x_1, y_1) and (x_2, y_2) is
Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example:
Exercise:

Problem: Use the Distance Formula to find the distance between the points $(-5, -3)$ and $(7, 2)$.

Solution:

Write the Distance Formula. $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Label the points, $\left(\overset{x_1, y_1}{-5, -3}\right), \left(\overset{x_2, y_2}{7, 2}\right)$ and substitute. $$d = \sqrt{(7 - (-5))^2 + (2 - (-3))^2}$$

Simplify.
$$d = \sqrt{12^2 + 5^2}$$
$$d = \sqrt{144 + 25}$$
$$d = \sqrt{169}$$
$$d = 13$$

Note:
Exercise:

Problem: Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(5, 7)$.

Solution:

$d = 15$

Note:
Exercise:

Problem: Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-14, -10)$.

Solution:

$d = 13$

Example:
Exercise:

Problem:

Use the Distance Formula to find the distance between the points $(10, -4)$ and $(-1, 5)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution:

Write the Distance Formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Label the points, $\left(\overset{x_1, y_1}{10, -4}\right), \left(\overset{x_2, y_2}{-1, 5}\right)$ and substitute.	$d = \sqrt{(-1 - 10)^2 + (5 - (-4))^2}$
Simplify.	$d = \sqrt{(-11)^2 + 9^2}$ $d = \sqrt{121 + 81}$ $d = \sqrt{202}$
Since 202 is not a perfect square, we can leave the answer in exact form or find a decimal approximation.	$d = \sqrt{202}$ or $d \approx 14.2$

Note:
Exercise:

Problem:

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(3, 4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution:

$d = \sqrt{130}, d \approx 11.4$

Note:
Exercise:

Problem:

Use the Distance Formula to find the distance between the points $(-2,-5)$ and $(-3,-4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution:

$d=\sqrt{2}, d\approx 1.4$

Use the Midpoint Formula

It is often useful to be able to find the midpoint of a segment. For example, if you have the endpoints of the diameter of a circle, you may want to find the center of the circle which is the midpoint of the diameter. To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.

Note:
Midpoint Formula
The midpoint of the line segment whose endpoints are the two points (x_1,y_1) and (x_2,y_2) is
Equation:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.

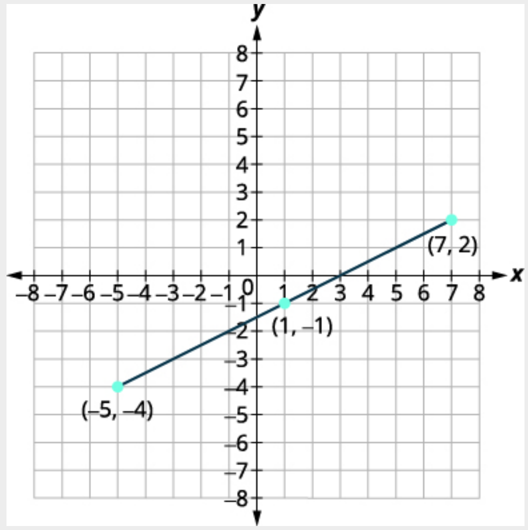
Example:
Exercise:

Problem:

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-5,-4)$ and $(7,2)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Solution:

Write the Midpoint Formula.	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$
Label the points, $\left(\overset{x_1,y_1}{-5,-4}\right),\left(\overset{x_2,y_2}{7,2}\right)$ and substitute.	$\left(\frac{-5+7}{2},\frac{-4+2}{2}\right)$
Simplify.	$\left(\frac{2}{2},\frac{-2}{2}\right)$

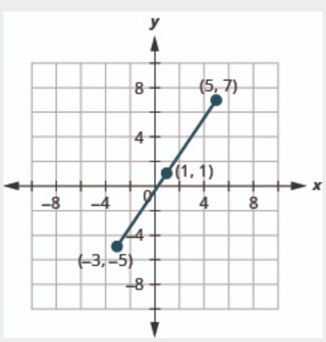
	$(1, -1)$ The midpoint of the segment is the point $(1, -1)$.
Plot the endpoints and midpoint.	

Note:
Exercise:

Problem:

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-3, -5)$ and $(5, 7)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Solution:

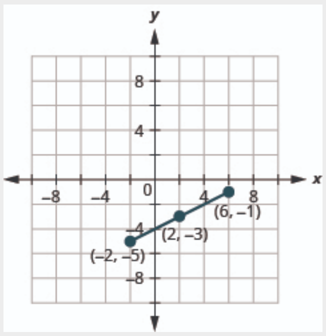


Note:
Exercise:

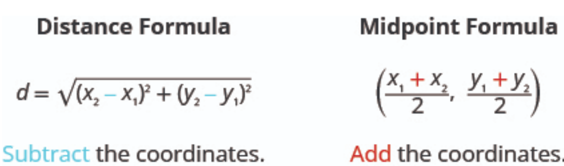
Problem:

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-2, -5)$ and $(6, -1)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Solution:

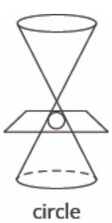


Both the Distance Formula and the Midpoint Formula depend on two points, (x_1, y_1) and (x_2, y_2). It is easy to confuse which formula requires addition and which subtraction of the coordinates. If we remember where the formulas come from, is may be easier to remember the formulas.



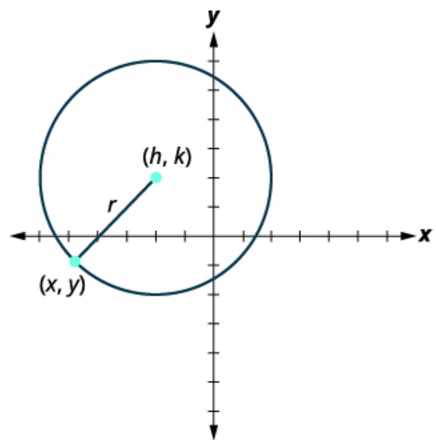
Write the Equation of a Circle in Standard Form

As we mentioned, our goal is to connect the geometry of a conic with algebra. By using the coordinate plane, we are able to do this easily.



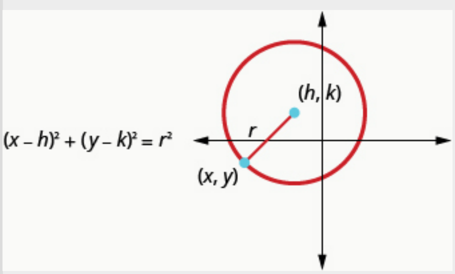
We define a circle as all points in a plane that are a fixed distance from a given point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius*, *r*, of the circle.

Note:
Circle
A circle is all points in a plane that are a fixed distance from a given point in the plane. The given point is called the **center**, (h, k), and the fixed distance is called the **radius**, *r*, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h,k), to a point on the circle, (x,y).	
To derive the equation of a circle, we can use the distance formula with the points (h,k), (x,y) and the distance, r.	$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Substitute the values.	$r=\sqrt{(x-h)^2+(y-k)^2}$
Square both sides.	$r^2=(x-h)^2+(y-k)^2$

This is the standard form of the equation of a circle with center, (h,k), and radius, r.

Note:
Standard Form of the Equation a Circle
The standard form of the equation of a circle with center, (h,k), and radius, r, is



Example:
Exercise:

Problem: Write the standard form of the equation of the circle with radius 3 and center $(0,0)$.

Solution:

Use the standard form of the equation of a circle	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values $r=3, h=0$, and $k=0$.	$(x-0)^2+(y-0)^2=3^2$
	
Simplify.	$x^2+y^2=9$

Note:
Exercise:

Problem: Write the standard form of the equation of the circle with a radius of 6 and center $(0,0)$.

Solution:

$x^2+y^2=36$

Note:
Exercise:

Problem: Write the standard form of the equation of the circle with a radius of 8 and center $(0,0)$.

Solution:

$x^2+y^2=64$

In the last example, the center was $(0,0)$. Notice what happened to the equation. Whenever the center is $(0,0)$, the standard form becomes $x^2+y^2=r^2$.

Example:
Exercise:

Problem: Write the standard form of the equation of the circle with radius 2 and center $(-1,3)$.

Solution:

Use the standard form of the equation of a	$(x-h)^2+(y-k)^2=r^2$

circle.	
Substitute in the values.	$(x-(-1))^2+(y-3)^2=2^2$
Center: $\left(\overset{h,\ k}{-1,\ 3}\right)$	
Simplify.	$(x+1)^2+(y-3)^2=4$

Note:
Exercise:

Problem: Write the standard form of the equation of the circle with a radius of 7 and center $(2,-4)$.

Solution:

$(x-2)^2+(y+4)^2=49$

Note:
Exercise:

Problem: Write the standard form of the equation of the circle with a radius of 9 and center $(-3,-5)$.

Solution:

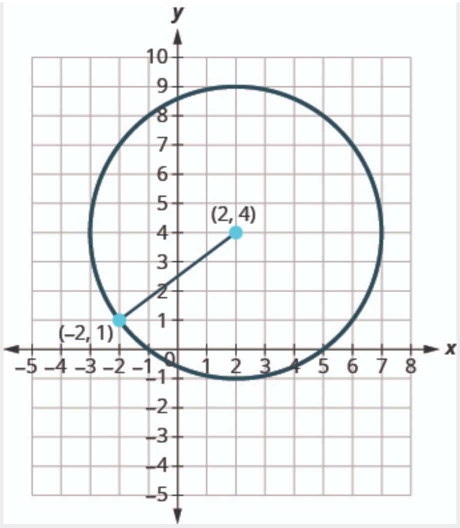
$(x+3)^2+(y+5)^2=81$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

Example:
Exercise:

Problem:

Write the standard form of the equation of the circle with center $(2,4)$ that also contains the point $(-2,1)$.



Solution:

The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2, 4)$ and point $(-2, 1)$

Use the Distance Formula to find the radius.	$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substitute the values. $\left(\overset{x_1, y_1}{2, 4}\right), \left(\overset{x_2, y_2}{-2, 1}\right)$	$r = \sqrt{(-2 - 2)^2 + (1 - 4)^2}$
Simplify.	$r = \sqrt{(-4)^2 + (-3)^2}$
	$r = \sqrt{16 + 9}$
	$r = \sqrt{25}$
	$r = 5$

Now that we know the radius, $r = 5$, and the center, $(2, 4)$, we can use the standard form of the equation of a circle to find the equation.

Use the standard form of the equation of a circle.	$(x - h)^2 + (y - k)^2$	=
Substitute in the values.	$(x - 2)^2 + (y - 4)^2$	$=$
Simplify.	$(x - 2)^2 + (y - 4)^2$	$=$

Note:
Exercise:

Problem:

Write the standard form of the equation of the circle with center $(2, 1)$ that also contains the point $(-2, -2)$.

Solution:

$(x - 2)^2 + (y - 1)^2 = 25$

Note:
Exercise:

Problem:

Write the standard form of the equation of the circle with center $(7,1)$ that also contains the point $(-1,-5)$.

Solution:

$(x-7)^2+(y-1)^2=100$

Graph a Circle

Any equation of the form $(x-h)^2+(y-k)^2=r^2$ is the standard form of the equation of a circle with center, (h,k), and radius, *r*. We can then graph the circle on a rectangular coordinate system.

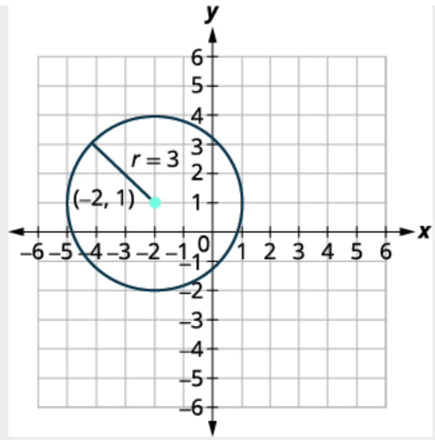
Note that the standard form calls for subtraction from *x* and *y*. In the next example, the equation has $x+2$, so we need to rewrite the addition as subtraction of a negative.

Example:
Exercise:

Problem: Find the center and radius, then graph the circle: $(x+2)^2+(y-1)^2=9$.

Solution:

	$(x+2)^2+(y-1)^2=9$
Use the standard form of the equation of a circle. Identify the center, (h,k) and radius, r.	$(x-h)^2+(y-k)^2=r^2$ $(x-(-2))^2+(y-1)^2=3^2$
	Center: $(-2,1)$ radius: 3
Graph the circle.	

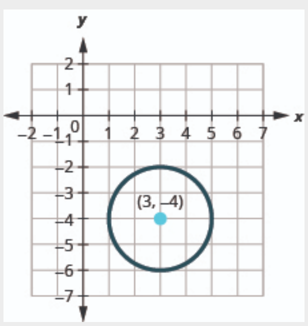


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $(x-3)^2+(y+4)^2=4.$

Solution:

ⓐ The circle is centered at $(3,-4)$ with a radius of 2.
ⓑ

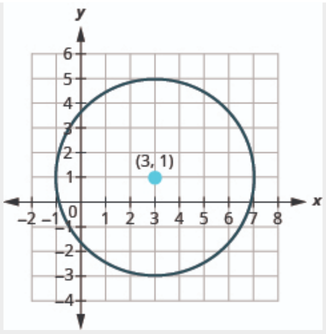


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $(x-3)^2+(y-1)^2=16.$

Solution:

ⓐ The circle is centered at $(3,1)$ with a radius of 4.
ⓑ



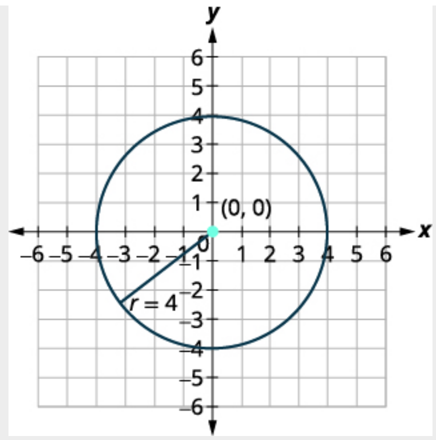
To find the center and radius, we must write the equation in standard form. In the next example, we must first get the coefficient of x^2, y^2 to be one.

Example:
Exercise:

Problem: Find the center and radius and then graph the circle, $4x^2 + 4y^2 = 64$.

Solution:

	$4x^2 + 4y^2 = 64$
Divide each side by 4.	$x^2 + y^2 = 16$
Use the standard form of the equation of a circle. Identify the center, (h, k) and radius, r.	$(x - h)^2 + (y - k)^2 = r^2$ $(x - 0)^2 + (y - 0)^2 = 4^2$
	Center: $(0, 0)$ radius: 4
Graph the circle.	

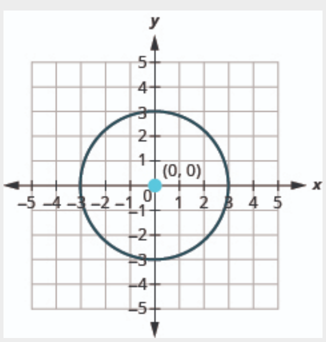


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $3x^2 + 3y^2 = 27$

Solution:

ⓐ The circle is centered at $(0, 0)$ with a radius of 3.
ⓑ

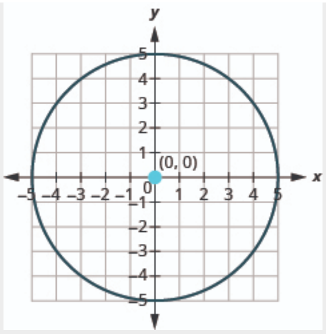


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $5x^2 + 5y^2 = 125$

Solution:

ⓐ The circle is centered at $(0, 0)$ with a radius of 5.
ⓑ



If we expand the equation from [link], $(x+2)^2+(y-1)^2=9$, the equation of the circle looks very different.

	$(x+2)^2+(y-1)^2 = 9$
Square the binomials.	$x^2+4x+4+y^2-2y+1 = 9$
Arrange the terms in descending degree order, and get zero on the right	$x^2+y^2+4x-2y-4 = 0$

This form of the equation is called the general form of the equation of the circle.

Note:
General Form of the Equation of a Circle
The general form of the equation of a circle is
Equation:

$$x^2+y^2+ax+by+c=0$$

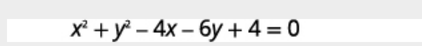
If we are given an equation in general form, we can change it to standard form by completing the squares in both *x* and *y*. Then we can graph the circle using its center and radius.

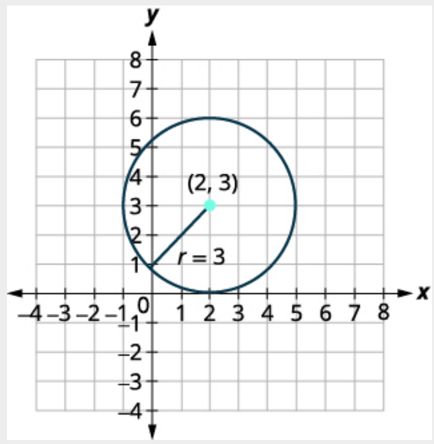
Example:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2+y^2-4x-6y+4=0$.

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.



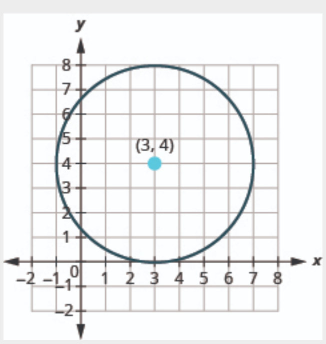
Group the *x*-terms and *y*-terms. Collect the constants on the right side.	$x^2 - 4x + y^2 - 6y = -4$
Complete the squares.	$x^2 - 4x + 4 + y^2 - 6y + 9 = -4 + 4 + 9$
Rewrite as binomial squares.	$(x-2)^2 + (y-3)^2 = 9$
Identify the center and radius.	Center: $(2, 3)$ radius: 3
Graph the circle.	

Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 - 6x - 8y + 9 = 0$.

Solution:

ⓐ The circle is centered at $(3, 4)$ with a radius of 4.
ⓑ

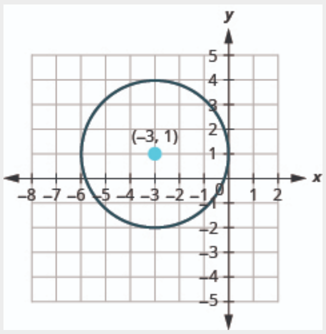


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 + 6x - 2y + 1 = 0$.

Solution:

ⓐ The circle is centered at $(-3, 1)$ with a radius of 3.
ⓑ



In the next example, there is a *y*-term and a y^2-term. But notice that there is no *x*-term, only an x^2-term. We have seen this before and know that it means *h* is 0. We will need to complete the square for the *y* terms, but not for the *x* terms.

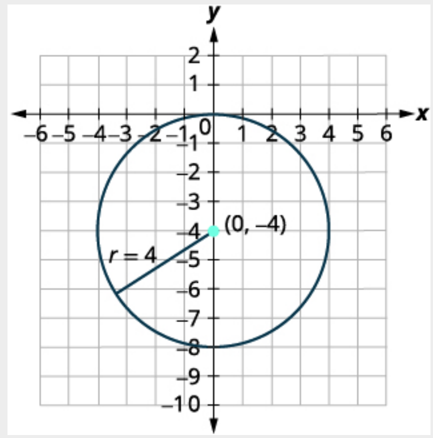
Example:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2 + y^2 + 8y = 0$.

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

	$x^2 + y^2 + 8y = 0$
Group the *x*-terms and *y*-terms.	$x^2 + y^2 + 8y = 0$
There are no constants to collect on the right side.	
Complete the square for $y^2 + 8y$.	$x^2 + y^2 + 8y + 16 = 0 + 16$

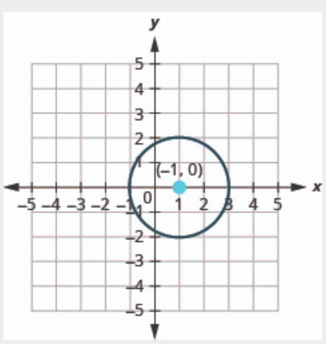
Rewrite as binomial squares.	$(x-0)^2+(y+4)^2=16$
Identify the center and radius.	Center: $(0,-4)$ radius: 4
Graph the circle.	

Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2+y^2-2x-3=0$.

Solution:

ⓐ The circle is centered at $(-1,0)$ with a radius of 2.

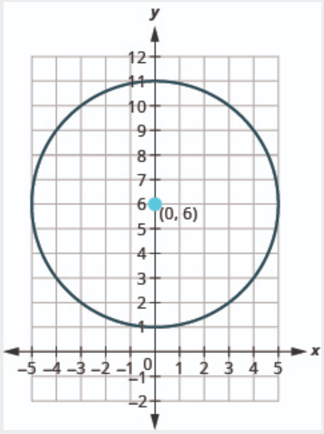


Note:
Exercise:

Problem: ⓐ Find the center and radius, then ⓑ graph the circle: $x^2+y^2-12y+11=0$.

Solution:

ⓐ The circle is centered at $(0, 6)$ with a radius of 5.



Note:
Access these online resources for additional instructions and practice with using the distance and midpoint formulas, and graphing circles.

- Distance-Midpoint Formulas and Circles
- Finding the Distance and Midpoint Between Two Points
- Completing the Square to Write Equation in Standard Form of a Circle

Key Concepts

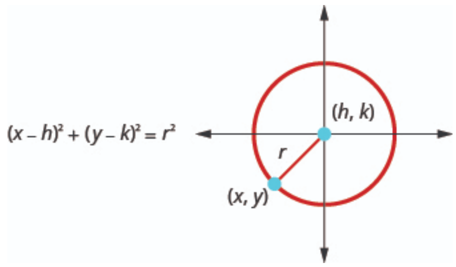
- **Distance Formula:** The distance *d* between the two points (x_1, y_1) and (x_2, y_2) is
  Equation:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Midpoint Formula:** The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is
  Equation:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  To find the midpoint of a line segment, we find the average of the *x*-coordinates and the average of the *y*-coordinates of the endpoints.
- **Circle:** A circle is all points in a plane that are a fixed distance from a fixed point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius, r,* of the circle.
- **Standard Form of the Equation a Circle:** The standard form of the equation of a circle with center, (h, k), and radius, *r,* is



- **General Form of the Equation of a Circle:** The general form of the equation of a circle is
  Equation:

$$x^2 + y^2 + ax + by + c = 0$$

Practice Makes Perfect

Use the Distance Formula

In the following exercises, find the distance between the points. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.
Exercise:

Problem: $(2, 0)$ and $(5, 4)$

Solution:

$d = 5$

Exercise:

Problem: $(-4, -3)$ and $(2, 5)$

Exercise:

Problem: $(-4, -3)$ and $(8, 2)$

Solution:

13

Exercise:

Problem: $(-7, -3)$ and $(8, 5)$

Exercise:

Problem: $(-1, 4)$ and $(2, 0)$

Solution:

5

Exercise:

Problem: $(-1, 3)$ and $(5, -5)$

Exercise:

Problem: $(1, -4)$ and $(6, 8)$

Solution:

13

Exercise:

Problem: $(-8, -2)$ and $(7, 6)$

Exercise:

Problem: $(-3, -5)$ and $(0, 1)$

Solution:

$76.\ d = 3\sqrt{5}, d \approx 6.7$

Exercise:

Problem: $(-1, -2)$ and $(-3, 4)$

Exercise:

Problem: $(3, -1)$ and $(1, 7)$

Solution:

$d = \sqrt{68}, d \approx 8.2$

Exercise:

Problem: $(-4, -5)$ and $(7, 4)$

Use the Midpoint Formula

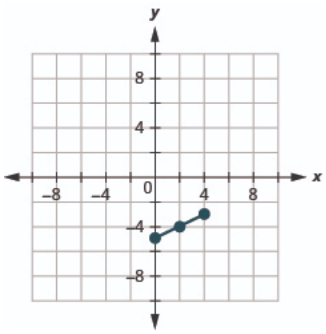
In the following exercises, ⓐ find the midpoint of the line segments whose endpoints are given and ⓑ plot the endpoints and the midpoint on a rectangular coordinate system.

Exercise:

Problem: $(0, -5)$ and $(4, -3)$

Solution:

ⓐ Midpoint: $(2, -4)$
ⓑ



Exercise:

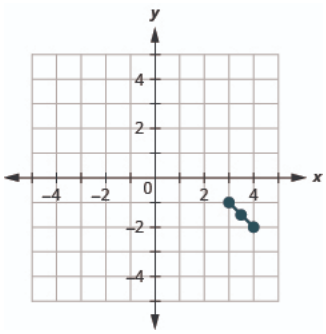
Problem: $(-2, -6)$ and $(6, -2)$

Exercise:

Problem: $(3, -1)$ and $(4, -2)$

Solution:

ⓐ Midpoint: $\left(3\frac{1}{2}, -1\frac{1}{2}\right)$
ⓑ



Exercise:

Problem: $(-3, -3)$ and $(6, -1)$

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0, 0)$.

Exercise:

Problem: Radius: 7

Solution:

$x^2 + y^2 = 49$

Exercise:

Problem: Radius: 9

Exercise:

Problem: Radius: $\sqrt{2}$

Solution:

$x^2 + y^2 = 2$

Exercise:

Problem: Radius: $\sqrt{5}$

In the following exercises, write the standard form of the equation of the circle with the given radius and center

Exercise:

Problem: Radius: 1, center: $(3, 5)$

Solution:

$(x - 3)^2 + (y - 5)^2 = 1$

Exercise:

Problem: Radius: 10, center: $(-2, 6)$

Exercise:

Problem: Radius: 2.5, center: $(1.5, -3.5)$

Solution:

$(x - 1.5)^2 + (y + 3.5)^2 = 6.25$

Exercise:

Problem: Radius: 1.5, center: $(-5.5, -6.5)$

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

Exercise:

Problem: Center $(3, -2)$ with point $(3, 6)$

Solution:

$(x - 3)^2 + (y + 2)^2 = 64$

Exercise:

Problem: Center $(6, -6)$ with point $(2, -3)$

Exercise:

Problem: Center $(4,4)$ with point $(2,2)$

Solution:

$(x-4)^2+(y-4)^2=8$

Exercise:

Problem: Center $(-5,6)$ with point $(-2,3)$

Graph a Circle

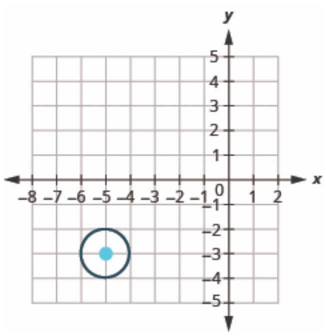
In the following exercises, ⓐ find the center and radius, then ⓑ graph each circle.

Exercise:

Problem: $(x+5)^2+(y+3)^2=1$

Solution:

ⓐ The circle is centered at $(-5,-3)$ with a radius of 1.
ⓑ



Exercise:

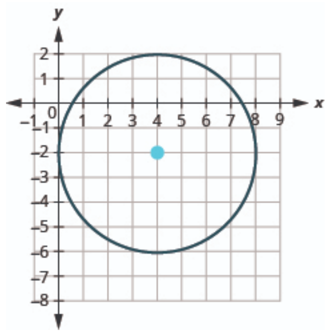
Problem: $(x-2)^2+(y-3)^2=9$

Exercise:

Problem: $(x-4)^2+(y+2)^2=16$

Solution:

ⓐ The circle is centered at $(4,-2)$ with a radius of 4.
ⓑ



Exercise:

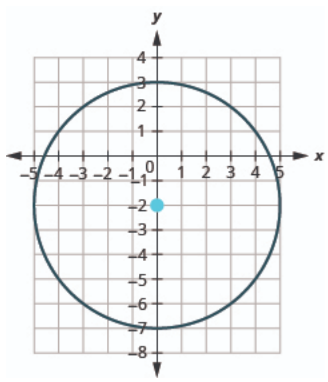
Problem: $(x+2)^2+(y-5)^2=4$

Exercise:

Problem: $x^2+(y+2)^2=25$

Solution:

ⓐ The circle is centered at $(0,-2)$ with a radius of 5.
ⓑ



Exercise:

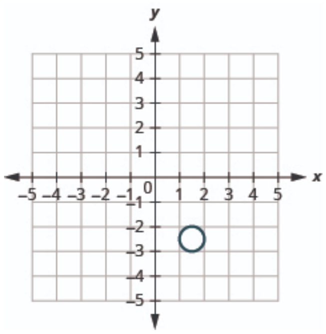
Problem: $(x-1)^2+y^2=36$

Exercise:

Problem: $(x-1.5)^2+(y+2.5)^2=0.25$

Solution:

ⓐ The circle is centered at $(1.5,2.5)$ with a radius of 0.5.
ⓑ



Exercise:

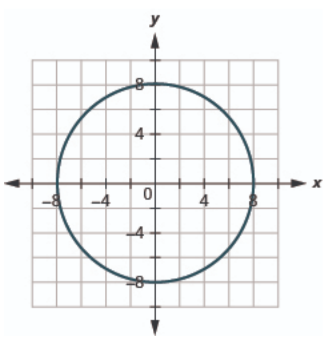
Problem: $(x-1)^2+(y-3)^2=\frac{9}{4}$

Exercise:

Problem: $x^2+y^2=64$

Solution:

ⓐ The circle is centered at $(0,0)$ with a radius of 8.
ⓑ



Exercise:

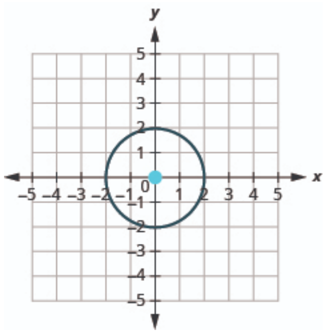
Problem: $x^2+y^2=49$

Exercise:

Problem: $2x^2+2y^2=8$

Solution:

ⓐ The circle is centered at $(0,0)$ with a radius of 2.
ⓑ



Exercise:

Problem: $6x^2 + 6y^2 = 216$

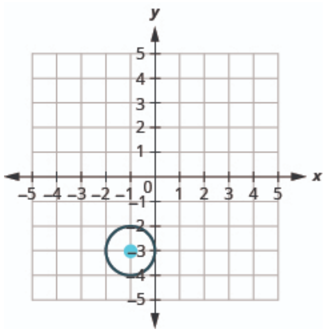
In the following exercises, ⓐ identify the center and radius and ⓑ graph.

Exercise:

Problem: $x^2 + y^2 + 2x + 6y + 9 = 0$

Solution:

ⓐ Center: $(-1, -3)$, radius: 1
ⓑ



Exercise:

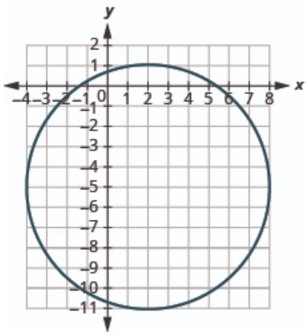
Problem: $x^2 + y^2 - 6x - 8y = 0$

Exercise:

Problem: $x^2 + y^2 - 4x + 10y - 7 = 0$

Solution:

ⓐ Center: $(2, -5)$, radius: 6
ⓑ



Exercise:

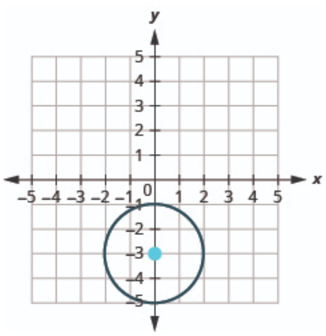
Problem: $x^2 + y^2 + 12x - 14y + 21 = 0$

Exercise:

Problem: $x^2 + y^2 + 6y + 5 = 0$

Solution:

ⓐ Center: $(0, -3)$, radius: 2
ⓑ



Exercise:

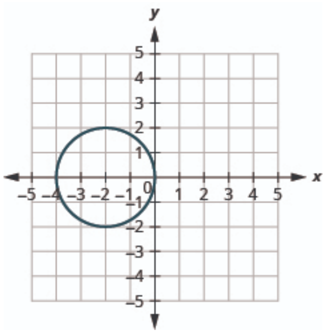
Problem: $x^2 + y^2 - 10y = 0$

Exercise:

Problem: $x^2 + y^2 + 4x = 0$

Solution:

ⓐ Center: $(-2, 0)$, radius:
ⓑ



Exercise:

Problem: $x^2 + y^2 - 14x + 13 = 0$

Writing Exercises

Exercise:

Problem: Explain the relationship between the distance formula and the equation of a circle.

Solution:

Answers will vary.

Exercise:

Problem: Is a circle a function? Explain why or why not.

Exercise:

Problem: In your own words, state the definition of a circle.

Solution:

Answers will vary.

Exercise:

Problem:

In your own words, explain the steps you would take to change the general form of the equation of a circle to the standard form.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance formula.			
use the midpoint formula.			
write the equation of a circle in standard form.			
graph a circle.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

circle
: A circle is all points in a plane that are a fixed distance from a fixed point in the plane.

Solve Systems of Nonlinear Equations
By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

Note:

1. Solve the system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$.
   If you missed this problem, review [link].
2. Solve the system by substitution: $\begin{cases} x - 4y = -4 \\ -3x + 4y = 0 \end{cases}$.
   If you missed this problem, review [link].
3. Solve the system by elimination: $\begin{cases} 3x - 4y = -9 \\ 5x + 3y = 14 \end{cases}$.
   If you missed this problem, review [link].

Solve a System of Nonlinear Equations Using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.
Equation:

$$\begin{cases} x^2 + y^2 = 9 \\ x^2 - y = 9 \end{cases} \qquad \begin{cases} 9x^2 + y^2 = 9 \\ y = 3x - 3 \end{cases} \qquad \begin{cases} x + y = 4 \\ y = x^2 + 2 \end{cases}$$

Note:
System of Nonlinear Equations
A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.

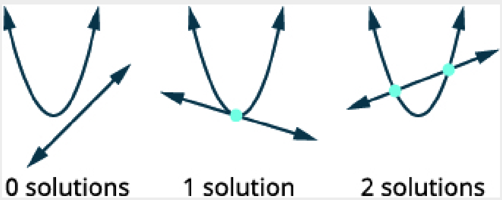
Note:
Solve a system of nonlinear equations by graphing.

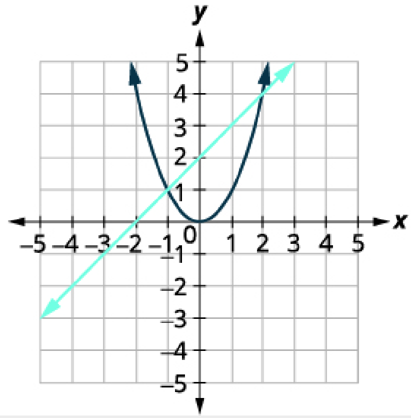
Identify the graph of each equation. Sketch the possible options for intersection.
Graph the first equation.
Graph the second equation on the same rectangular coordinate system.
Determine whether the graphs intersect.
Identify the points of intersection.
Check that each ordered pair is a solution to both original equations.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x - y = -2 \\ y = x^2 \end{cases}$.

Solution:

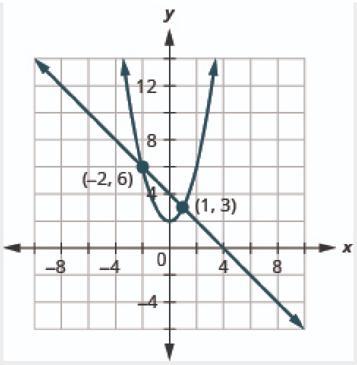
Identify each graph.	$\begin{cases} x - y = -2 & \text{line} \\ y = x^2 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
Graph the line, $x - y = -2$. Slope-intercept form $y = x + 2$. Graph the parabola, $y = x^2$.	

	
Identify the points of intersection.	The points of intersection appear to be $(2,4)$ and $(-1,1)$.
Check to make sure each solution makes both equations true. $(2,4)$ $x - y = -2 \quad y = x^2$ $2 - 4 \stackrel{?}{=} -2 \quad 4 \stackrel{?}{=} 2^2$ $-2 = -2 \checkmark \quad 4 = 4 \checkmark$ $(-1,1)$ $x - y = -2 \quad y = x^2$ $-1 - 1 \stackrel{?}{=} -2 \quad 1 \stackrel{?}{=} (-1)^2$ $-2 = -2 \checkmark \quad 1 = 1 \checkmark$	
	The solutions are $(2,4)$ and $(-1,1)$.

Note:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x + y = 4 \\ y = x^2 + 2 \end{cases}$.

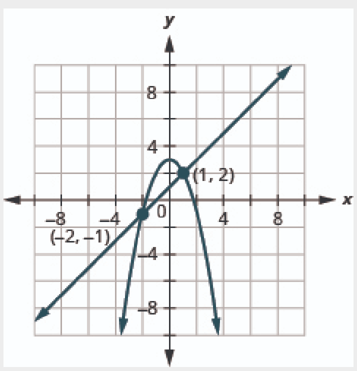
Solution:



Note:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x - y = -1 \\ y = -x^2 + 3 \end{cases}$.

Solution:

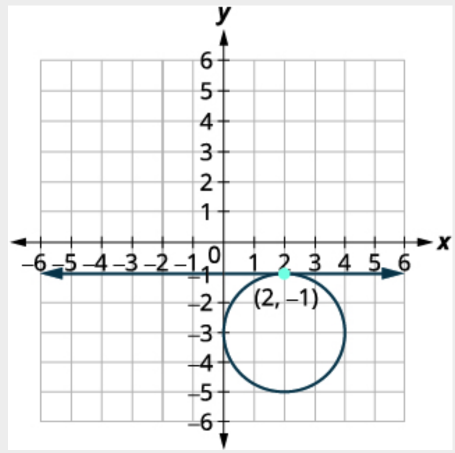


To identify the graph of each equation, keep in mind the characteristics of the x^2 and y^2 terms of each conic.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = -1 \\ (x-2)^2 + (y+3)^2 = 4 \end{cases}$.

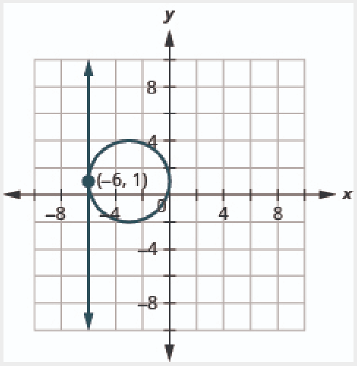
Solution:

Identify each graph.	$\begin{cases} y = -1 & \text{line} \\ (x-2)^2 + (y+3)^2 = 4 & \text{circle} \end{cases}$
Sketch the possible options for the intersection of a circle and a line.	
Graph the circle, $(x-2)^2 + (y+3)^2 = 4$ Center: $(2, -3)$ radius: 2 Graph the line, $y = -1$. It is a horizontal line.	
Identify the points of intersection.	The point of intersection appears to be $(2, -1)$.
Check to make sure the solution makes both equations true. $(2, -1)$ $\begin{aligned} (x-2)^2 + (y+3)^2 &= 4 \\ (2-2)^2 + (-1+3)^2 &\stackrel{?}{=} 4 \\ (0)^2 + (2)^2 &\stackrel{?}{=} 4 \\ 4 &= 4 \checkmark \end{aligned}$ $\begin{aligned} y &= -1 \\ -1 &= -1 \checkmark \end{aligned}$	
	The solution is $(2, -1)$.

Note:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x = -6 \\ (x+3)^2 + (y-1)^2 = 9 \end{cases}$.

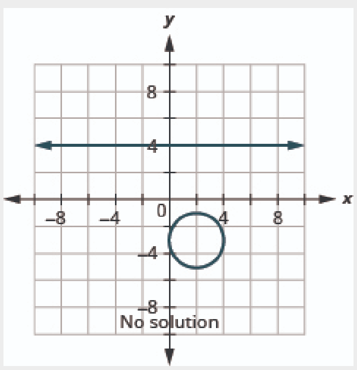
Solution:



Note:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = 4 \\ (x-2)^2 + (y+3)^2 = 4 \end{cases}$.

Solution:



Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.

Note:
Solve a system of nonlinear equations by substitution.

Identify the graph of each equation. Sketch the possible options for intersection.
Solve one of the equations for either variable.
Substitute the expression from Step 2 into the other equation.
Solve the resulting equation.
Substitute each solution in Step 4 into one of the original equations to find the other variable.
Write each solution as an ordered pair.
Check that each ordered pair is a solution to**both**original equations.

Example:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} 9x^2 + y^2 = 9 \\ y = 3x - 3 \end{cases}$.

Solution:

Identify each graph.	$\begin{cases} 9x^2 + y^2 = 9 & \text{ellipse} \\ y = 3x - 3 & \text{line} \end{cases}$
Sketch the possible options for intersection of an ellipse and a line.	
The equation $y = 3x - 3$ is solved for y.	$y = 3x - 3$
	$9x^2 + y^2 = 9$
Substitute $3x - 3$ for y in the first equation.	$9x^2 + (3x - 3)^2 = 9$

Solve the equation for *x*.	$9x^2 + 9x^2 - 18x + 9 = 9$
	$18x^2 - 18x = 0$ $18x(x - 1) = 0$ $x = 0 \quad x = 1$
Substitute $x = 0$ and $x = 1$ into $y = 3x - 3$ to find *y*.	$y = 3x - 3 \qquad y = 3x - 3$
	$y = 3 \cdot 0 - 3 \qquad y = 3 \cdot 1 - 3$ $y = -3 \qquad y = 0$
	The ordered pairs are $(0, -3)$, $(1, 0)$.
Check **both** ordered pairs in **both** equations. $(0, -3)$ $\begin{array}{rcl} 9x^2 + y^2 & = & 9 \\ 9 \cdot 0^2 + (-3)^2 & \stackrel{?}{=} & 9 \\ 0 + 9 & \stackrel{?}{=} & 9 \\ 9 & = & 9\checkmark \end{array} \qquad \begin{array}{rcl} y & = & 3x - 3 \\ -3 & \stackrel{?}{=} & 3 \cdot 0 - 3 \\ -3 & \stackrel{?}{=} & 0 - 3 \\ -3 & = & -3\checkmark \end{array}$ $(1, 0)$ $\begin{array}{rcl} 9x^2 + y^2 & = & 9 \\ 9 \cdot 1^2 + 0^2 & \stackrel{?}{=} & 9 \\ 9 + 0 & \stackrel{?}{=} & 9 \\ 9 & = & 9\checkmark \end{array} \qquad \begin{array}{rcl} y & = & 3x - 3 \\ 0 & \stackrel{?}{=} & 3 \cdot 1 - 3 \\ 0 & \stackrel{?}{=} & 3 - 3 \\ 0 & = & 0\checkmark \end{array}$	
	The solutions are $(0, -3), (1, 0)$.

Note:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} x^2 + 9y^2 = 9 \\ y = \frac{1}{3}x - 3 \end{cases}$.

Solution:

No solution

Note:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} 4x^2 + y^2 = 4 \\ y = x + 2 \end{cases}$.

Solution:

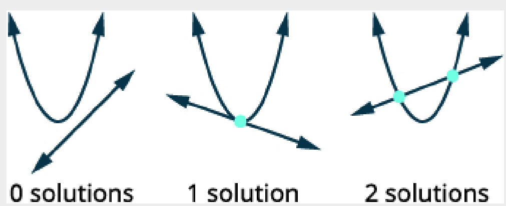
$\left(-\frac{4}{5}, \frac{6}{5}\right), (0, 2)$

So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

Example:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = x - 2 \end{cases}$.

Solution:

Identify each graph.	$\begin{cases} x^2 - y = 0 & \text{parabola} \\ y = x - 2 & \text{line} \end{cases}$
Sketch the possible options for intersection of a parabola and a line	
The equation $y = x - 2$ is solved for y.	$y = x - 2$

	$x^2 - y = 0$
Substitute $x - 2$ for y in the first equation.	$x^2 - (x - 2) = 0$
Solve the equation for x.	$x^2 - x + 2 = 0$
This doesn't factor easily, so we can check the discriminant.	
$b^2 - 4ac$ $(-1)^2 - 4 \cdot 1 \cdot 2$ -7	The discriminant is negative, so there is no real solution. The system has no solution.

Note:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = 2x - 3 \end{cases}$.

Solution:

No solution

Note:
Exercise:

Problem: Solve the system by using substitution: $\begin{cases} y^2 - x = 0 \\ y = 3x - 2 \end{cases}$.

Solution:

$\left(\frac{4}{9}, -\frac{2}{3}\right), (1, 1)$

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations

have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of linear equations. The steps are listed for reference.

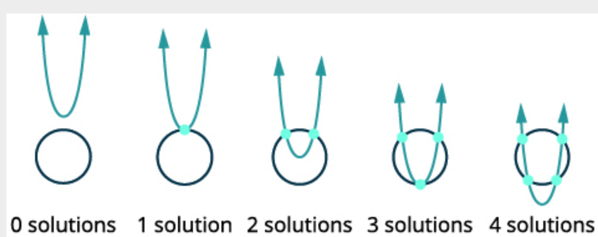
Note:
Solve a system of equations by elimination.

Identify the graph of each equation. Sketch the possible options for intersection.
Write both equations in standard form.
Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.
Add the equations resulting from Step 3 to eliminate one variable.
Solve for the remaining variable.
Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
Write each solution as an ordered pair.
Check that each ordered pair is a solution to **both** original equations.

Example:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 4 \end{cases}$.

Solution:

Identify each graph.	$\begin{cases} x^2 + y^2 = 4 & \text{circle} \\ x^2 - y \;\; = 4 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a circle and a parabola.	

Both equations are in standard form.	$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y \;\;= 4 \end{cases}$
To get opposite coefficients of x^2, we will multiply the second equation by -1.	$\begin{cases} x^2 + y^2 = 4 \\ -1(x^2 - y) \;\;= -1(4) \end{cases}$
Simplify.	$\begin{cases} x^2 + y^2 = 4 \\ -x^2 + y \;\;= 4 \end{cases}$
Add the two equations to eliminate x^2.	$\begin{cases} x^2 + y^2 = 4 \\ -x^2 + y \;\;= 4 \end{cases}$ $y^2 + y \;\;= 0$
Solve for *y*.	$y(y+1) = 0$
	$y = 0 \quad y + 1 = 0$ $y = -1$
Substitute $y = 0$ and $y = -1$ into one of the original equations. Then solve for *x*.	$y = 0 \quad y = -1$
	$x^2 - y \;\;= 4 \qquad x^2 - y \;\;= 4$ $x^2 - 0 \;\;= 4 \qquad x^2 - (-1) = 4$ $x^2 = 4 \qquad x^2 = 3$ $x = \pm 2 \qquad x = \pm\sqrt{3}$
Write each solution as an ordered pair.	The ordered pairs are $(-2, 0)$ $(2, 0)$. $\left(\sqrt{3}, -1\right)$ $\left(-\sqrt{3}, -1\right)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2, 0)$, $(2, 0)$, $\left(\sqrt{3}, -1\right)$, and $\left(-\sqrt{3}, -1\right)$.

Note:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$.

Solution:

$(-3,0),(3,0),\left(-2\sqrt{2},-1\right),\left(2\sqrt{2},-1\right)$

Note:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$.

Solution:

$(-1,0),(0,1),(0,-1)$

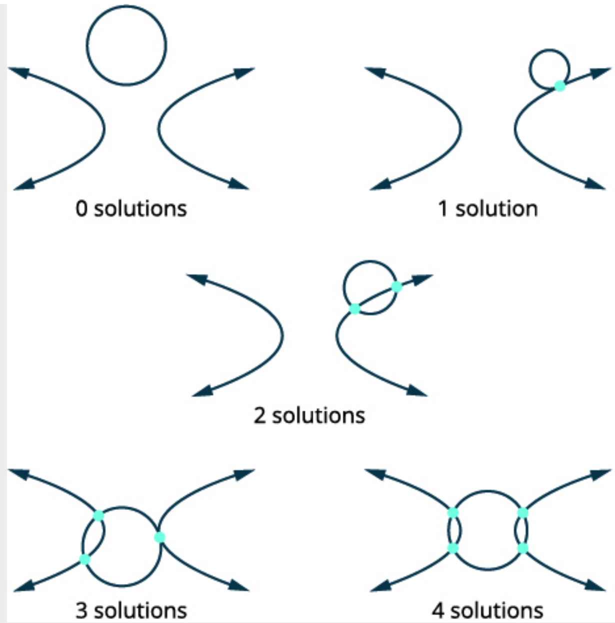
There are also four options when we consider a circle and a hyperbola.

Example:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$.

Solution:

Identify each graph.	$\begin{cases} x^2+y^2=7 & \text{circle} \\ x^2-y^2=1 & \text{hyperbola} \end{cases}$
Sketch the possible options for intersection of a circle and hyperbola.	

	
Both equations are in standard form.	$\begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 1 \end{cases}$
The coefficients of y^2 are opposite, so we will add the equations.	$\underline{\begin{cases} x^2 + y^2 = 7 \\ x^2 - y^2 = 1 \end{cases}}$ $2x^2 \qquad = 8$
Simplify.	$x^2 = 4$ $x = \pm 2$ $x = 2 \qquad x = -2$
Substitute $x = 2$ and $x = -2$ into one of the original equations. Then solve for y.	$\begin{aligned} x^2 + y^2 &= 7 \\ 2^2 + y^2 &= 7 \\ 4 + y^2 &= 7 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$ $\qquad$ $\begin{aligned} x^2 + y^2 &= 7 \\ (-2)^2 + y^2 &= 7 \\ 4 + y^2 &= 7 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$
Write each solution as an ordered pair.	The ordered pairs are $\left(-2, \sqrt{3}\right)$, $\left(-2, -\sqrt{3}\right)$, $\left(2, \sqrt{3}\right)$, and $\left(2, -\sqrt{3}\right)$.
Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four	The solutions are $\left(-2, \sqrt{3}\right)$, $\left(-2, -\sqrt{3}\right)$,

solutions to you.	$\left(2, \sqrt{3}\right)$, and $\left(2, -\sqrt{3}\right)$.

Note:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2 + y^2 = 25 \\ y^2 - x^2 = 7 \end{cases}$.

Solution:

$(-3, -4), (-3, 4), (3, -4), (3, 4)$

Note:
Exercise:

Problem: Solve the system by elimination: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 4 \end{cases}$.

Solution:

$(-2, 0), (2, 0)$

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

Example:
Exercise:

Problem:

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Solution:

Identify what we are looking for.	Two different numbers.
Define the variables.	$x =$ first number $y =$ second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.
	$x^2 - y^2 = 15$
Second sentence.	The sum of the numbers is 5.
	$x + y = 5$
Solve the system by substitution	$\begin{cases} x^2 - y^2 = 15 \\ x + y = 5 \end{cases}$
Solve the second equation for *x*.	$x = 5 - y$
Substitute *x* into the first equation.	$x^2 - y^2 = 15$
	$(5 - y)^2 - y^2 = 15$
Expand and simplify.	$(25 - 10y + y^2) - y^2 = 15$
	$25 - 10y + y^2 - y^2 = 15$ $25 - 10y = 15$
Solve for *y*.	$-10y = -10$
	$y = 1$

Substitute back into the second equation.	$x + y = 5$
	$x + (1) = 5$ $x = 4$
	The numbers are 1 and 4.

Note:
Exercise:

Problem:

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

Solution:

4 and 6

Note:
Exercise:

Problem:

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

Solution:

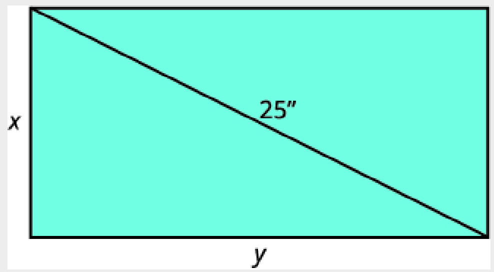
-18 and 17

Example:
Exercise:

Problem:

Myra purchased a small 25” TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Solution:

Identify what we are looking for.	The length and width of the rectangle
Define the variables.	Let $x =$ width of the rectangle $y =$ length of the rectangle
Draw a diagram to help visualize the situation.	
	Area is 300 square inches.
Translate the information into a system of equations.	The diagonal of the right triangle is 25 inches.
	$x^2 + y^2 = 25^2$ $x^2 + y^2 = 625$
	The area of the rectangle is 300 square inches.
	$x \cdot y = 300$ $\begin{cases} x^2 + y^2 = 625 \\ x \cdot y = 300 \end{cases}$
Solve the system using substitution.	$x \cdot y = 300$
Solve the second equation for x.	$x = \frac{300}{y}$
Substitute x into the first equation.	$x^2 + y^2 = 625$
	$\left(\frac{300}{y}\right)^2 + y^2 = 625$
Simplify.	

	$\frac{90000}{y^2} + y^2 = 625$
Multiply by y^2 to clear the fractions.	$90000 + y^4 = 625y^2$
Put in standard form.	$y^4 - 625y^2 + 90000 = 0$
Solve by factoring.	$(y^2 - 225)(y^2 - 400) = 0$
	$y^2 - 225 = 0 \quad y^2 - 400 = 0$
	$y^2 = 225 \quad y^2 = 400$ $y = \pm 15 \quad y = \pm 20$
Since y is a side of the rectangle, we discard the negative values.	$y = 15 \quad y = 20$
Substitute back into the second equation.	$x \cdot y = 300 \quad x \cdot y = 300$
	$x \cdot 15 = 300 \quad x \cdot 20 = 300$ $x = 20 \quad x = 15$
	If the length is 15 inches, the width is 20 inches.
	If the length is 20 inches, the width is 15 inches.

Note:
Exercise:

Problem:

Edgar purchased a small 20” TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and width of the TV screen?

Solution:

If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

Note:
Exercise:

Problem:

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

Solution:

If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

Note:
Access these online resources for additional instructions and practice with solving nonlinear equations.

- Nonlinear Systems of Equations
- Solve a System of Nonlinear Equations
- Solve a System of Nonlinear Equations by Elimination
- System of Nonlinear Equations – Area and Perimeter Application

Key Concepts

- **How to solve a system of nonlinear equations by graphing.**

  Identify the graph of each equation. Sketch the possible options for intersection.
  Graph the first equation.
  Graph the second equation on the same rectangular coordinate system.
  Determine whether the graphs intersect.
  Identify the points of intersection.
  Check that each ordered pair is a solution to both original equations.

- **How to solve a system of nonlinear equations by substitution.**

  Identify the graph of each equation. Sketch the possible options for intersection.
  Solve one of the equations for either variable.
  Substitute the expression from Step 2 into the other equation.
  Solve the resulting equation.
  Substitute each solution in Step 4 into one of the original equations to find the other variable.
  Write each solution as an ordered pair.
  Check that each ordered pair is a solution to **both** original equations.

- **How to solve a system of equations by elimination.**

  Identify the graph of each equation. Sketch the possible options for intersection.
  Write both equations in standard form.
  Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.
  Add the equations resulting from Step 3 to eliminate one variable.
  Solve for the remaining variable.
  Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
  Write each solution as an ordered pair.
  Check that each ordered pair is a solution to**both**original equations.

Section Exercises

Practice Makes Perfect

Solve a System of Nonlinear Equations Using Graphing

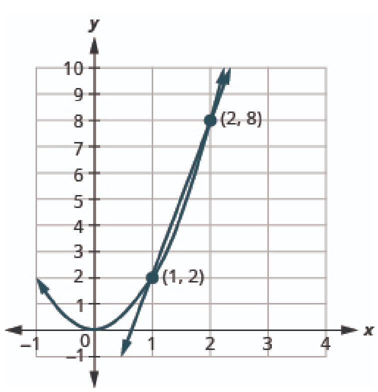
In the following exercises, solve the system of equations by using graphing.
Exercise:

Problem: $\begin{cases} y = 2x + 2 \\ y = -x^2 + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 6x - 4 \\ y = 2x^2 \end{cases}$

Solution:



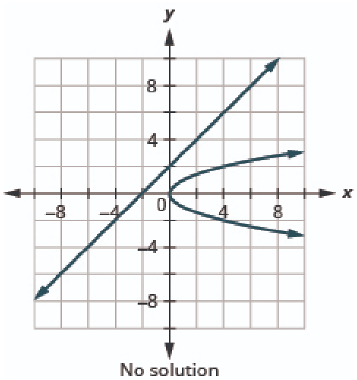
Exercise:

Problem: $\begin{cases} x + y = 2 \\ x = y^2 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y = -2 \\ x = y^2 \end{cases}$

Solution:



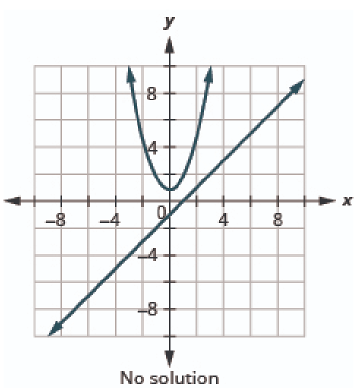
Exercise:

Problem: $\begin{cases} y = \frac{3}{2}x + 3 \\ y = -x^2 + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} y = x - 1 \\ y = x^2 + 1 \end{cases}$

Solution:



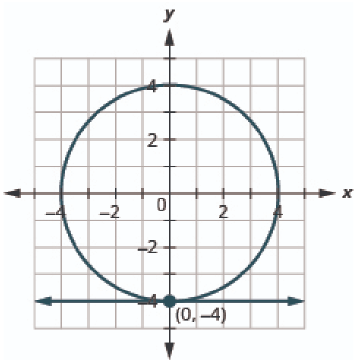
Exercise:

Problem: $\begin{cases} x = -2 \\ x^2 + y^2 = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -4 \\ x^2 + y^2 = 16 \end{cases}$

Solution:



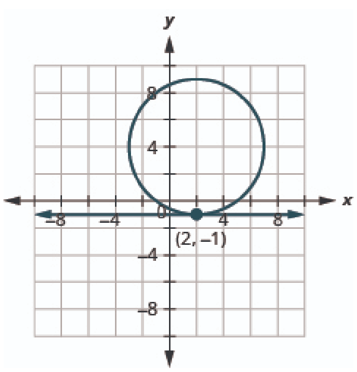
Exercise:

Problem: $\begin{cases} x = 2 \\ (x+2)^2 + (y+3)^2 = 16 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -1 \\ (x-2)^2 + (y-4)^2 = 25 \end{cases}$

Solution:



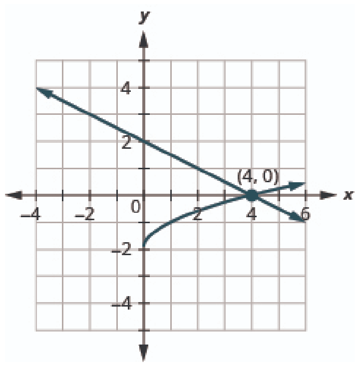
Exercise:

Problem: $\begin{cases} y = -2x + 4 \\ y = \sqrt{x} + 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -\frac{1}{2}x + 2 \\ y = \sqrt{x} - 2 \end{cases}$

Solution:



Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

Exercise:

Problem: $\begin{cases} x^2 + 4y^2 = 4 \\ y = \frac{1}{2}x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 9x^2 + y^2 = 9 \\ y = 3x + 3 \end{cases}$

Solution:

$(-1, 0), (0, 3)$

Exercise:

Problem: $\begin{cases} 9x^2 + y^2 = 9 \\ y = x + 3 \end{cases}$

Exercise:

Problem: $\begin{cases} 9x^2 + 4y^2 = 36 \\ x = 2 \end{cases}$

Solution:

$(2, 0)$

Exercise:

Problem: $\begin{cases} 4x^2 + y^2 = 4 \\ y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 169 \\ x = 12 \end{cases}$

Solution:

$(12, -5), (12, 5)$

Exercise:

Problem: $\begin{cases} 3x^2 - y = 0 \\ y = 2x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 2y^2 - x = 0 \\ y = x + 1 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} y = x^2 + 3 \\ y = x + 3 \end{cases}$

Exercise:

Problem: $\begin{cases} y = x^2 - 4 \\ y = x - 4 \end{cases}$

Solution:

$(0, -4), (1, -3)$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 25 \\ x - y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

Solution:

$(3, 4), (5, 0)$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y = 8 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 16 \\ x^2 - y = 4 \end{cases}$

Solution:

$(0, -4), \left(-\sqrt{7}, 3\right), \left(\sqrt{7}, 3\right)$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 4 \\ x^2 + 2y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 2 \end{cases}$

Solution:

$(0, -2), \left(-\sqrt{3}, 1\right), \left(\sqrt{3}, 1\right)$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 4 \\ y^2 - x = 2 \end{cases}$

Solution:

$(-2, 0), \left(1, -\sqrt{3}\right), \left(1, \sqrt{3}\right)$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 25 \\ 2x^2 - 3y^2 = 5 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 20 \\ x^2 - y^2 = -12 \end{cases}$

Solution:

$(-2, -4), (-2, 4), (2, -4), (2, 4)$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 13 \\ x^2 - y^2 = 5 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 16 \\ x^2 - y^2 = 16 \end{cases}$

Solution:

$(-4, 0), (4, 0)$

Exercise:

Problem: $\begin{cases} 4x^2 + 9y^2 = 36 \\ 2x^2 - 9y^2 = 18 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 - y^2 = 3 \\ 2x^2 + y^2 = 6 \end{cases}$

Solution:

$\left(-\sqrt{3}, 0\right), \left(\sqrt{3}, 0\right)$

Exercise:

Problem: $\begin{cases} 4x^2 - y^2 = 4 \\ 4x^2 + y^2 = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 - y^2 = -5 \\ 3x^2 + 2y^2 = 30 \end{cases}$

Solution:

$(-2, -3), (-2, 3), (2, -3), (2, 3)$

Exercise:

Problem: $\begin{cases} x^2 - y^2 = 1 \\ x^2 - 2y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x^2 + y^2 = 11 \\ x^2 + 3y^2 = 28 \end{cases}$

Solution:

$(-1, -3), (-1, 3), (1, -3), (1, 3)$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

Exercise:

Problem: The sum of two numbers is -6 and the product is 8. Find the numbers.

Exercise:

Problem: The sum of two numbers is 11 and the product is -42. Find the numbers.

Solution:

-3 and 14

Exercise:

Problem:

The sum of the squares of two numbers is 65. The difference of the number is 3. Find the numbers.

Exercise:

Problem:

The sum of the squares of two numbers is 113. The difference of the number is 1. Find the numbers.

Solution:

-7 and -8 or 8 and 7

Exercise:

Problem:

The difference of the squares of two numbers is 15. The difference of twice the square of the first number and the square of the second number is 30. Find the numbers.

Exercise:

Problem:

The difference of the squares of two numbers is 20. The difference of the square of the first number and twice the square of the second number is 4. Find the numbers.

Solution:

-6 and -4 or -6 and 4 or 6 and -4 or 6 and 4

Exercise:

Problem:

The perimeter of a rectangle is 32 inches and its area is 63 square inches. Find the length and width of the rectangle.

Exercise:

Problem:

The perimeter of a rectangle is 52 cm and its area is 165 cm^2. Find the length and width of the rectangle.

Solution:

If the length is 11 cm, the width is 15 cm. If the length is 15 cm, the width is 11 cm.

Exercise:

Problem:

Dion purchased a new microwave. The diagonal of the door measures 17 inches. The door also has an area of 120 square inches. What are the length and width of the microwave door?

Exercise:

Problem:

Jules purchased a microwave for his kitchen. The diagonal of the front of the microwave measures 26 inches. The front also has an area of 240 square inches. What are the length and width of the microwave?

Solution:

If the length is 10 inches, the width is 24 inches. If the length is 24 inches, the width is 10 inches.

Exercise:

Problem:

Roman found a widescreen TV on sale, but isn't sure if it will fit his entertainment center. The TV is 60". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1728 square inches. His entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Roman's entertainment center?

Exercise:

Problem:

Donnette found a widescreen TV at a garage sale, but isn't sure if it will fit her entertainment center. The TV is 50". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1200 square inches. Her entertainment center has an insert for the TV with a length of 38 inches and width of 27 inches. What are the length and width of the TV screen and will it fit Donnette's entertainment center?

Solution:

The length is 40 inches and the width is 30 inches. The TV will not fit Donnette's entertainment center.

Writing Exercises

Exercise:

Problem:

In your own words, explain the advantages and disadvantages of solving a system of equations by graphing.

Exercise:

Problem: Explain in your own words how to solve a system of equations using substitution.

Solution:

Answers will vary.

Exercise:

Problem: Explain in your own words how to solve a system of equations using elimination.

Exercise:

Problem:

A circle and a parabola can intersect in ways that would result in 0, 1, 2, 3, or 4 solutions. Draw a sketch of each of the possibilities.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a system of nonlinear equations using graphing.			
solve a system of nonlinear equations using substitution.			
solve a system of nonlinear equations using elimination.			
use a system of nonlinear equations to solve applications.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

Distance and Midpoint Formulas; Circles

Use the Distance Formula

In the following exercises, find the distance between the points. Round to the nearest tenth if needed.
Exercise:

Problem: $(-5, 1)$ and $(-1, 4)$

Exercise:

Problem: $(-2, 5)$ and $(1, 5)$

Solution:

$d = 3$

Exercise:

Problem: $(8, 2)$ and $(-7, -3)$

Exercise:

Problem: $(1, -4)$ and $(5, -5)$

Solution:

$d = \sqrt{17}, d \approx 4.1$

Use the Midpoint Formula

In the following exercises, find the midpoint of the line segments whose endpoints are given.

Exercise:

Problem: $(-2, -6)$ and $(-4, -2)$

Exercise:

Problem: $(3, 7)$ and $(5, 1)$

Solution:

$(4, 4)$

Exercise:

Problem: $(-8, -10)$ and $(9, 5)$

Exercise:

Problem: $(-3, 2)$ and $(6, -9)$

Solution:

$\left(\frac{3}{2}, -\frac{7}{2}\right)$

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given information.

Exercise:

Problem: radius is 15 and center is $(0, 0)$

Exercise:

Problem: radius is $\sqrt{7}$ and center is $(0, 0)$

Solution:

$x^2 + y^2 = 7$

Exercise:

Problem: radius is 9 and center is $(-3, 5)$

Exercise:

Problem: radius is 7 and center is $(-2, -5)$

Solution:

$(x + 2)^2 + (y + 5)^2 = 49$

Exercise:

Problem: center is $(3, 6)$ and a point on the circle is $(3, -2)$

Exercise:

Problem: center is $(2, 2)$ and a point on the circle is $(4, 4)$

Solution:

$(x - 2)^2 + (y - 2)^2 = 8$

Graph a Circle

In the following exercises, ⓐ find the center and radius, then ⓑ graph each circle.

Exercise:

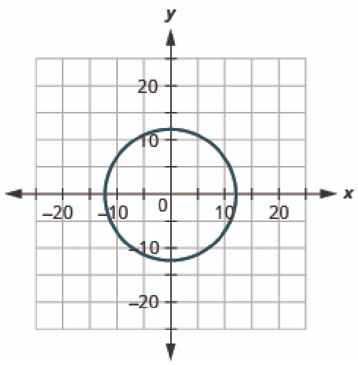
Problem: $2x^2 + 2y^2 = 450$

Exercise:

Problem: $3x^2 + 3y^2 = 432$

Solution:

ⓐ radius: 12, center: $(0, 0)$
ⓑ



Exercise:

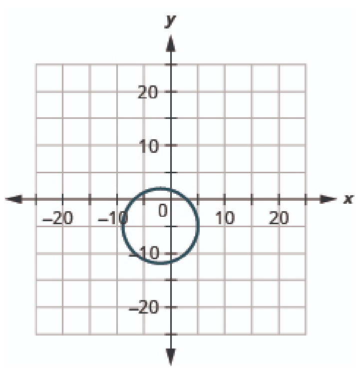
Problem: $(x+3)^2+(y-5)^2=81$

Exercise:

Problem: $(x+2)^2+(y+5)^2=49$

Solution:

ⓐ radius: 7, center: $(-2,-5)$
ⓑ



Exercise:

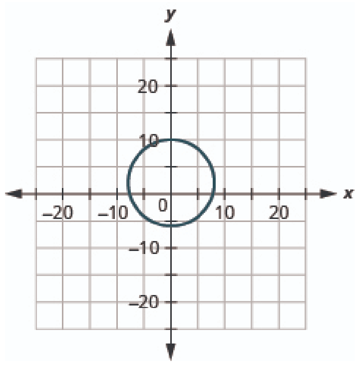
Problem: $x^2+y^2-6x-12y-19=0$

Exercise:

Problem: $x^2+y^2-4y-60=0$

Solution:

ⓐ radius: 8, center: $(0,2)$
ⓑ



Parabolas

Graph Vertical Parabolas

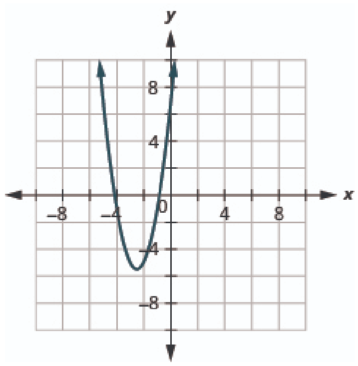
In the following exercises, graph each equation by using its properties.
Exercise:

Problem: $y = x^2 + 4x - 3$

Exercise:

Problem: $y = 2x^2 + 10x + 7$

Solution:



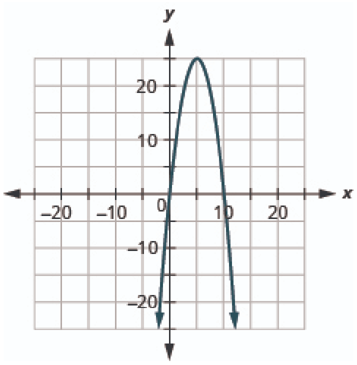
Exercise:

Problem: $y = -6x^2 + 12x - 1$

Exercise:

Problem: $y = -x^2 + 10x$

Solution:



In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

Exercise:

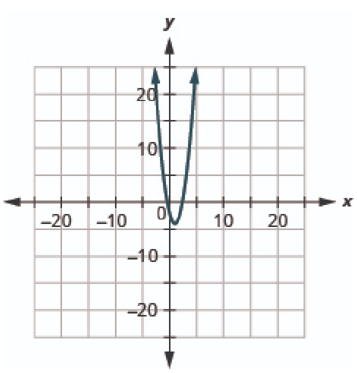
Problem: $y = x^2 + 4x + 7$

Exercise:

Problem: $y = 2x^2 - 4x - 2$

Solution:

ⓐ $y = 2(x-1)^2 - 4$
ⓑ



Exercise:

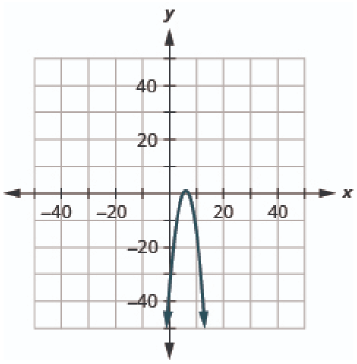
Problem: $y = -3x^2 - 18x - 29$

Exercise:

Problem: $y = -x^2 + 12x - 35$

Solution:

ⓐ $y=-(x-6)^2+1$
ⓑ



Graph Horizontal Parabolas

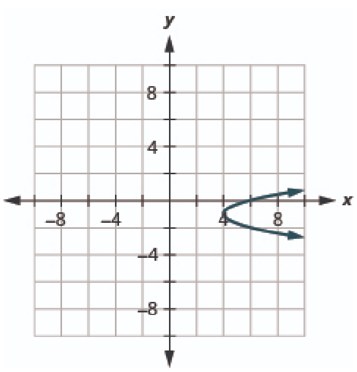
In the following exercises, graph each equation by using its properties.
Exercise:

Problem: $x=2y^2$

Exercise:

Problem: $x=2y^2+4y+6$

Solution:



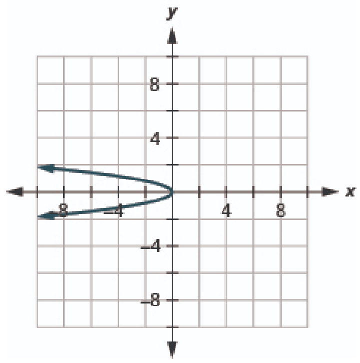
Exercise:

Problem: $x=-y^2+2y-4$

Exercise:

Problem: $x=-3y^2$

Solution:



In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

Exercise:

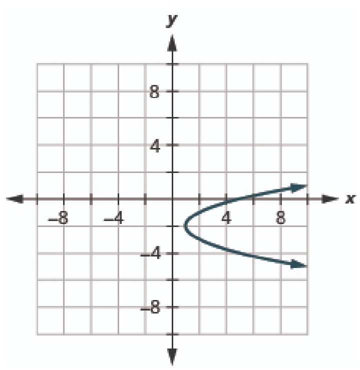
Problem: $x = 4y^2 + 8y$

Exercise:

Problem: $x = y^2 + 4y + 5$

Solution:

ⓐ $x = (y + 2)^2 + 1$
ⓑ



Exercise:

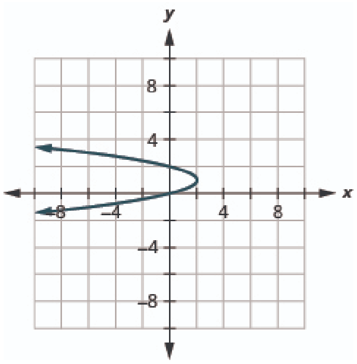
Problem: $x = -y^2 - 6y - 7$

Exercise:

Problem: $x = -2y^2 + 4y$

Solution:

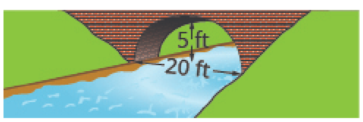
ⓐ $x=-2(y-1)^2+2$
ⓑ



Solve Applications with Parabolas

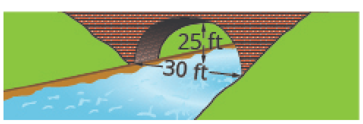
In the following exercises, create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in standard form.
Exercise:

Problem:



Exercise:

Problem:



Solution:

$y=-\frac{1}{9}x^2+\frac{10}{3}x$

Ellipses

Graph an Ellipse with Center at the Origin

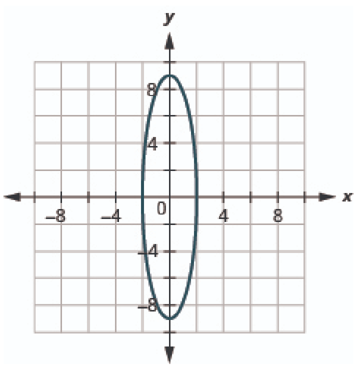
In the following exercises, graph each ellipse.
Exercise:

Problem: $\frac{x^2}{36}+\frac{y^2}{25}=1$

Exercise:

Problem: $\frac{x^2}{4} + \frac{y^2}{81} = 1$

Solution:



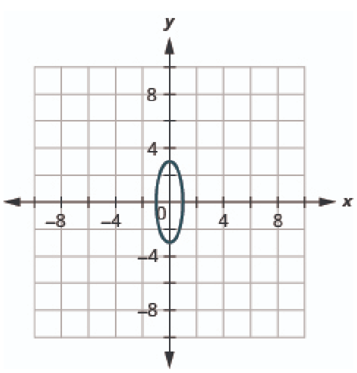
Exercise:

Problem: $49x^2 + 64y^2 = 3136$

Exercise:

Problem: $9x^2 + y^2 = 9$

Solution:

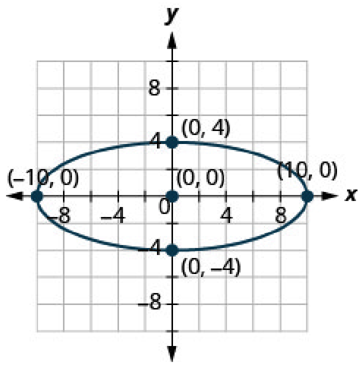


Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

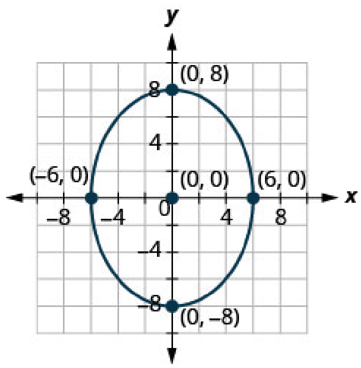
Exercise:

Problem:



Exercise:

Problem:



Solution:

$\frac{x^2}{36} + \frac{y^2}{64} = 1$

Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

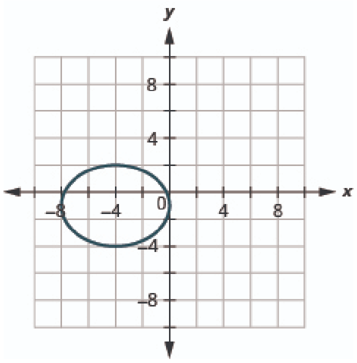
Exercise:

Problem: $\frac{(x-1)^2}{25} + \frac{(y-6)^2}{4} = 1$

Exercise:

Problem: $\frac{(x+4)^2}{16} + \frac{(y+1)^2}{9} = 1$

Solution:



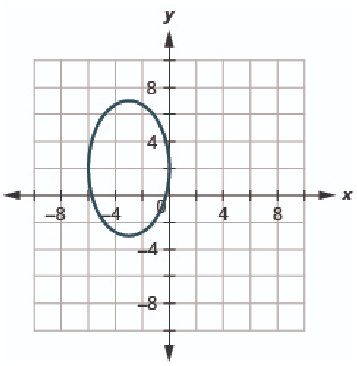
Exercise:

Problem: $\frac{(x-5)^2}{16}+\frac{(y+3)^2}{36}=1$

Exercise:

Problem: $\frac{(x+3)^2}{9}+\frac{(y-2)^2}{25}=1$

Solution:



In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

Exercise:

Problem: $x^2+y^2+12x+40y+120=0$

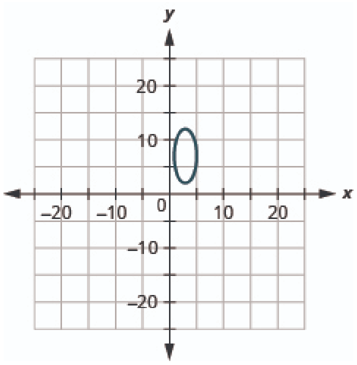
Exercise:

Problem: $25x^2+4y^2-150x-56y+321=0$

Solution:

ⓐ $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$

ⓑ



Exercise:

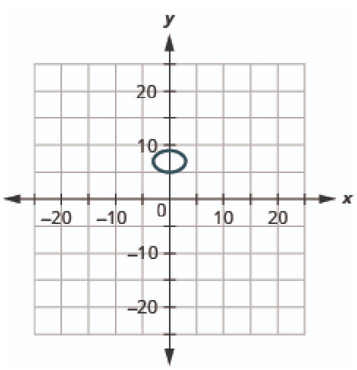
Problem: $25x^2 + 4y^2 + 150x + 125 = 0$

Exercise:

Problem: $4x^2 + 9y^2 - 126x + 405 = 0$

Solution:

ⓐ $\frac{x^2}{9} + \frac{(y-7)^2}{4} = 1$
ⓑ



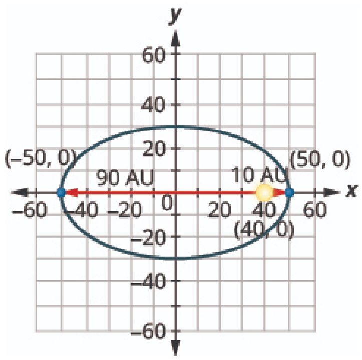
Solve Applications with Ellipses

In the following exercises, write the equation of the ellipse described.

Exercise:

Problem:

A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 10 AU and the furthest is approximately 90 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



Hyperbolas

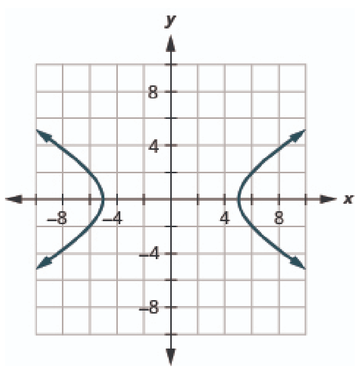
Graph a Hyperbola with Center at $(0, 0)$

In the following exercises, graph.

Exercise:

Problem: $\frac{x^2}{25} - \frac{y^2}{9} = 1$

Solution:



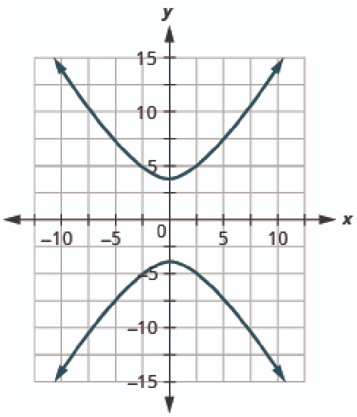
Exercise:

Problem: $\frac{y^2}{49} - \frac{x^2}{16} = 1$

Exercise:

Problem: $9y^2 - 16x^2 = 144$

Solution:



Exercise:

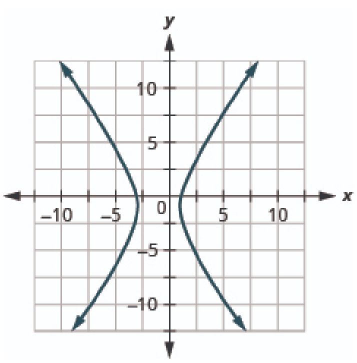
Problem: $16x^2 - 4y^2 = 64$

Graph a Hyperbola with Center at (h, k)

In the following exercises, graph.
Exercise:

Problem: $\frac{(x+1)^2}{4} - \frac{(y+1)^2}{9} = 1$

Solution:



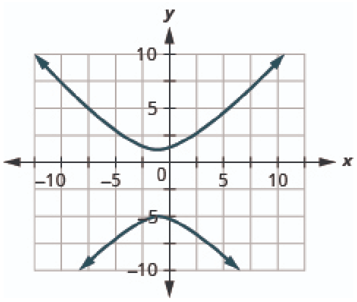
Exercise:

Problem: $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{16} = 1$

Exercise:

Problem: $\frac{(y+2)^2}{9} - \frac{(x+1)^2}{9} = 1$

Solution:



Exercise:

Problem: $\frac{(y-1)^2}{25} - \frac{(x-2)^2}{9} = 1$

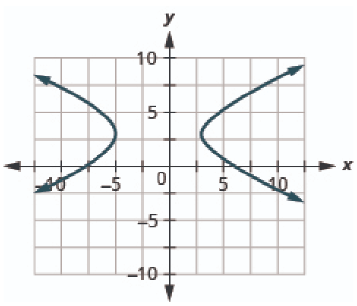
In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

Exercise:

Problem: $4x^2 - 16y^2 + 8x + 96y - 204 = 0$

Solution:

ⓐ $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{4} = 1$
ⓑ



Exercise:

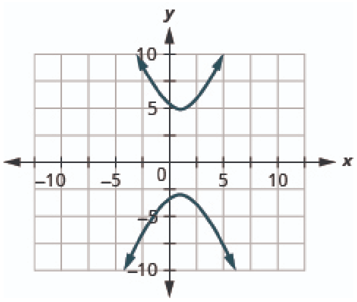
Problem: $16x^2 - 4y^2 - 64x - 24y - 36 = 0$

Exercise:

Problem: $4y^2 - 16x^2 + 32x - 8y - 76 = 0$

Solution:

ⓐ $\frac{(y-1)^2}{16} - \frac{(x-1)^2}{4} = 1$
ⓑ



Exercise:

Problem: $36y^2 - 16x^2 - 96x + 216y - 396 = 0$

Identify the Graph of each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

Exercise:

Problem: ⓐ $16y^2 - 9x^2 - 36x - 96y - 36 = 0$
ⓑ $x^2 + y^2 - 4x + 10y - 7 = 0$
ⓒ $y = x^2 - 2x + 3$
ⓓ $25x^2 + 9y^2 = 225$

Solution:

ⓐ hyperbola ⓑ circle ⓒ parabola ⓓ ellipse

Exercise:

Problem: ⓐ $x^2 + y^2 + 4x - 10y + 25 = 0$
ⓑ $y^2 - x^2 - 4y + 2x - 6 = 0$
ⓒ $x = -y^2 - 2y + 3$
ⓓ $16x^2 + 9y^2 = 144$

Solve Systems of Nonlinear Equations

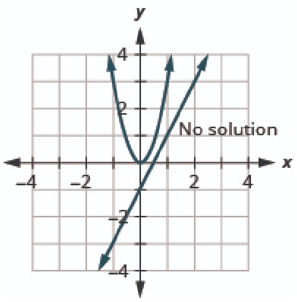
Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

Exercise:

Problem: $\begin{cases} 3x^2 - y = 0 \\ y = 2x - 1 \end{cases}$

Solution:



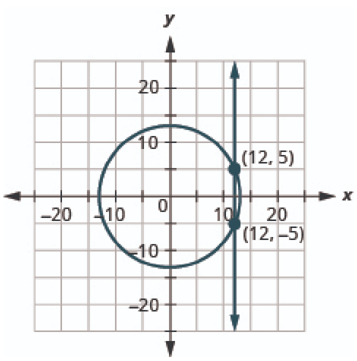
Exercise:

Problem: $\begin{cases} y = x^2 - 4 \\ y = x - 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x^2 + y^2 = 169 \\ x = 12 \end{cases}$

Solution:



Exercise:

Problem: $\begin{cases} x^2 + y^2 = 25 \\ y = -5 \end{cases}$

Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

Exercise:

Problem: $\begin{cases} y = x^2 + 3 \\ y = -2x + 2 \end{cases}$

Solution:

$(-1,4)$

Exercise:

Problem: $\begin{cases} x^2+y^2=4 \\ x-y=4 \end{cases}$

Exercise:

Problem: $\begin{cases} 9x^2+4y^2=36 \\ y-x=5 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} x^2+4y^2=4 \\ 2x-y=1 \end{cases}$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

Exercise:

Problem: $\begin{cases} x^2+y^2=16 \\ x^2-2y-1=0 \end{cases}$

Solution:

$\left(-\sqrt{7},3\right),\left(\sqrt{7},3\right)$

Exercise:

Problem: $\begin{cases} x^2-y^2=5 \\ -2x^2-3y^2=-30 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x^2+9y^2=36 \\ 3y^2-4x=12 \end{cases}$

Solution:

$(-3,0),(0,-2),(0,2)$

Exercise:

Problem: $\begin{cases} x^2+y^2=14 \\ x^2-y^2=16 \end{cases}$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.
Exercise:

Problem:

The sum of the squares of two numbers is 25. The difference of the numbers is 1. Find the numbers.

Solution:

-3 and -4 or 4 and 3

Exercise:

Problem:

The difference of the squares of two numbers is 45. The difference of the square of the first number and twice the square of the second number is 9. Find the numbers.

Exercise:

Problem:

The perimeter of a rectangle is 58 meters and its area is 210 square meters. Find the length and width of the rectangle.

Solution:

If the length is 14 inches, the width is 15 inches. If the length is 15 inches, the width is 14 inches.

Exercise:

Problem:

Colton purchased a larger microwave for his kitchen. The diagonal of the front of the microwave measures 34 inches. The front also has an area of 480 square inches. What are the length and width of the microwave?

Practice Test

In the following exercises, find the distance between the points and the midpoint of the line segment with the given endpoints. Round to the nearest tenth as needed.
Exercise:

Problem: $(-4,-3)$ and $(-10,-11)$

Solution:

distance: 10, midpoint: $(-7,-7)$

Exercise:

Problem: $(6, 8)$ and $(-5, -3)$

In the following exercises, write the standard form of the equation of the circle with the given information.

Exercise:

Problem: radius is 11 and center is $(0, 0)$

Solution:

$x^2 + y^2 = 121$

Exercise:

Problem: radius is 12 and center is $(10, -2)$

Exercise:

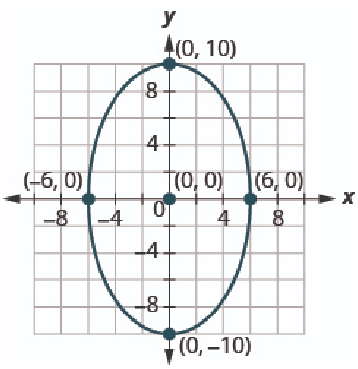
Problem: center is $(-2, 3)$ and a point on the circle is $(2, -3)$

Solution:

$(x + 2)^2 + (y - 3)^2 = 52$

Exercise:

Problem: Find the equation of the ellipse shown in the graph.



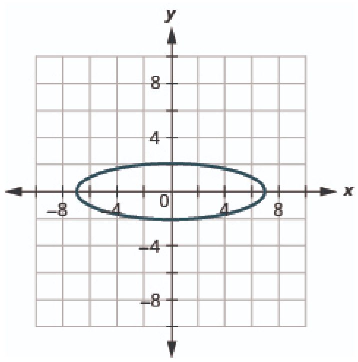
In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, and ⓑ graph the equation.

Exercise:

Problem: $4x^2 + 49y^2 = 196$

Solution:

ⓐ ellipse
ⓑ



Exercise:

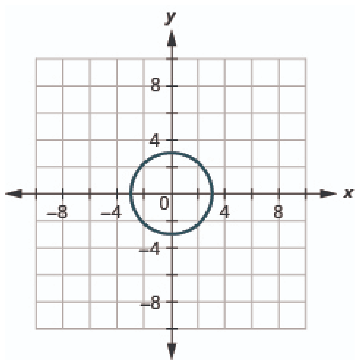
Problem: $y=3{(x-2)}^{2}-2$

Exercise:

Problem: $3{x}^{2}+3{y}^{2}=27$

Solution:

ⓐ circle
ⓑ



Exercise:

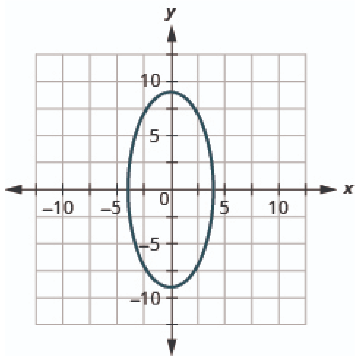
Problem: $\frac{{y}^{2}}{100}-\frac{{x}^{2}}{36}=1$

Exercise:

Problem: $\frac{{x}^{2}}{16}+\frac{{y}^{2}}{81}=1$

Solution:

ⓐ ellipse
ⓑ



Exercise:

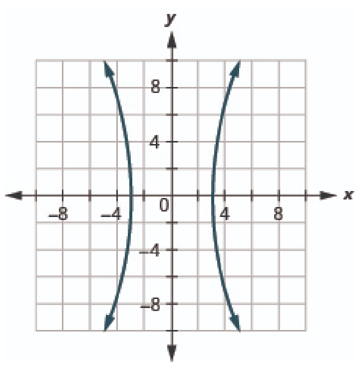
Problem: $x = 2y^2 + 10y + 7$

Exercise:

Problem: $64x^2 - 9y^2 = 576$

Solution:

ⓐ hyperbola
ⓑ



In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, ⓑ write the equation in standard form, and ⓒ graph the equation.

Exercise:

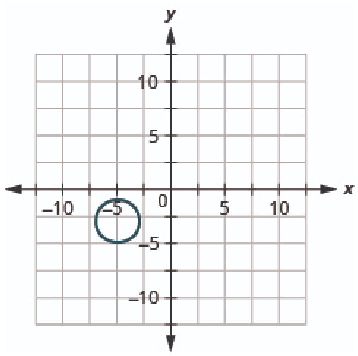
Problem: $25x^2 + 64y^2 + 200x - 256y - 944 = 0$

Exercise:

Problem: $x^2 + y^2 + 10x + 6y + 30 = 0$

Solution:

ⓐ circle
ⓑ $(x+5)^2+(y+3)^2=4$
ⓒ



Exercise:

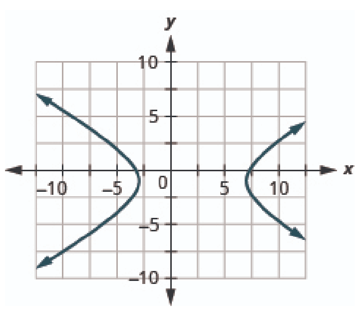
Problem: $x=-y^2+2y-4$

Exercise:

Problem: $9x^2-25y^2-36x-50y-214=0$

Solution:

ⓐ hyperbola
ⓑ $\frac{(x-2)^2}{25}-\frac{(y+1)^2}{9}=1$
ⓒ



Exercise:

Problem: $y=x^2+6x+8$

Exercise:

Problem: Solve the nonlinear system of equations by graphing: $\begin{cases} 3y^2 - x = 0 \\ y = -2x - 1 \end{cases}.$

Solution:

No solution

Exercise:

Problem: Solve the nonlinear system of equations using substitution: $\begin{cases} x^2 + y^2 = 8 \\ y = -x - 4 \end{cases}.$

Exercise:

Problem: Solve the nonlinear system of equations using elimination: $\begin{cases} x^2 + 9y^2 = 9 \\ 2x^2 - 9y^2 = 18 \end{cases}.$

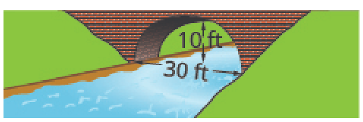
Solution:

$(0, -3), (0, 3)$

Exercise:

Problem:

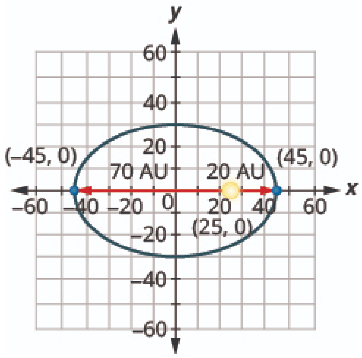
Create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in $y = ax^2 + bx + c$ form.



Exercise:

Problem:

A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 20 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



Solution:

$$\frac{x^2}{2025} + \frac{y^2}{1400} = 1$$

Exercise:

Problem: The sum of two numbers is 22 and the product is -240. Find the numbers.

Exercise:

Problem:

For her birthday, Olive's grandparents bought her a new widescreen TV. Before opening it she wants to make sure it will fit her entertainment center. The TV is 55". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1452 square inches. Her entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Olive's entertainment center?

Solution:

The length is 44 inches and the width is 33 inches. The TV will fit Olive's entertainment center.

Glossary

system of nonlinear equations
: A system of nonlinear equations is a system where at least one of the equations is not linear.